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PRACTICAL METHOD OF FLEXURAL STRENGTH CALCULATION OF REINFORCED AND PRESTRESSED CONCRETE MEMBERS

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Abstract. In this paper the flexural strength analysis of reinforced and prestressed concrete members with symmetrical cross-sections loaded in the plane of symmetry is performed. A new practical method for determining the height of the compression zone is proposed. The method is valid for normal and high-strength concretes and for different distributions of bars along the section. It is based on the assumptions, simplifications and material properties of Eurocode 2. Design equations have been developed for the rectangular stress distribution in the concrete compression zone and for the steel stress-strain diagrams with a horizontal and inclined top branch. A numerical example is presented to show the method usage.

Keywords: calculation method, flexural strength, reinforced and prestressed concrete, symmetrical cross-section, plane of bending, compression zone height.

1. Introduction

The flexural strength analysis of reinforced and prestressed concrete members is based on conditions of static equilibrium, strain compatibility and material properties. The unknown members in the corresponding equations can be expressed in simple or complicated way depending on the applied stress-strain curves for materials.

This paper observes symmetrical cross-sections loaded in the plane of symmetry. A detailed section analysis based on complete stress-strain relations can be carried out. In this case, the compression force in the concrete is calculated by the integration of stresses over the concrete area and the force equilibrium at the section is achieved by an iterative procedure. The integration methods and the iterative procedure are laborious to perform manually, therefore computers are used [1–3]. In spite of general use of computers the practical methods of calculation have not lost their importance in certain cases. These methods enable simpler calculations, control of computer results and overview of the performed computations. The practical methods are either graphical or analytical. The graphical interaction diagrams are usually available for simple sections (rectangle, circle), normal-strength concrete and more commonly used steel locations [4–6]. For high-strength concrete and uncommon arrangement of steel these diagrams cannot be applied. In the absence of specific interaction diagram, an alternative to construct the diagram is to use the analytical expressions. The complexity of expressions depends on the used idealisations for stress-strain curves and the geometry of section. It is very usual to apply the rectangular stress block assumption for concrete and the elastic–perfectly plastic approximation for reinforcing and prestressing steel. More commonly used cross-sections, for which at these idealisations the analytical expressions exist, are rectangular and flanged sections with a distribution of reinforcement along the sides parallel to the axis of bending. As the prestressing steel does not exhibit a well-defined yield plateau than in Eurocode 2 [7], the design is allowed to base on the actual curve, if this is known or in some other codes the analytical expressions for the shape of actual curve are accepted. In these cases, the corresponding flexural strength analysis leads to an iterative procedure [8], which for practical calculations can be tedious. Several other approximate procedures for the prestressing steel in the flexural strength analysis have been presented [9, 10].

The objective of the current paper is to present a practical method of calculation by which a symmetrical cross-section of beams and columns with arbitrary arrangement of bars can be analysed. The method can be applied to both normal and high-strength concretes and various stress-strain relationships for reinforcing and prestressing steel according to Eurocode 2.

2. Strain distribution in a cross-section

The strain distribution of cross-section will be found by using "plane sections remain plane" assumption and in Standard EN1992-1-1:2004 [7] given stress-strain relationships for materials. Two strains are needed to determine the strain distribution for members carrying load in the plane of symmetry. One of them can be the strain at neutral axis and the other one of the following limiting strains (Fig 1):

- reinforcing or prestressing steel tension strain limit ε_{ud} (point A);
- concrete compression strain limit ε_{cu2} or ε_{cu3} depending on the stress-strain diagram used (point B);
- concrete pure compression strain limit ε_{c2} or ε_{c3} (point C).

For cases where there is a compression over all the section of limiting strain ε_{c2} or ε_{c3} occurs in the fibre located at x_c from the extreme compression edge:

$$x_c = \left(1 - \frac{\varepsilon_{c2}}{\varepsilon_{cu2}}\right)h \text{ or } x_c = \left(1 - \frac{\varepsilon_{c3}}{\varepsilon_{cu3}}\right)h. \quad (1)$$

When the rotations in bending are small, the strains are proportional to the angle change over a unit length of beam ψ and vary linearly with distance y from the neutral axis:

$$\varepsilon = \psi y. \quad (2)$$

From the geometry in the figure (using limiting strains ε_{c2} and ε_{cu2}):

$$\psi = \begin{cases} \dfrac{\varepsilon_{cu2}}{x}, & \text{if } x \le h, \\ \dfrac{\varepsilon_{c2}}{x - x_c}, & \text{if } x > h, \\ \dfrac{\varepsilon_{ud}}{d - x}, & \text{if } x \le \dfrac{\varepsilon_{cu2}}{\varepsilon_{ud} + \varepsilon_{cu2}} d, \end{cases} \quad (3)$$

where x is the height of compression zone, d – the distance from the extreme compression fibre to the centroid of the tension steel and ε_{cu2}, ε_{c2}, ε_{ud} are the above-mentioned strains. The strains ε_{cu2}, ε_{c2}, ε_{ud} have been taken as positive.

The Eq (2) can be modified for calculating steel strains: the strain in the reinforcing steel ε_s:

$$\varepsilon_s = \begin{cases} \psi(d_s - x), & \text{if strains in steel are not limited} \\ \psi(d_s - x) \le \varepsilon_{ud}, & \text{if strains in steel are limited;} \end{cases} \quad (4)$$

the strain in the prestressing steel ε_p with the prestrain ε_{pm}:

$$\varepsilon_p = \begin{cases} \varepsilon_{pm} + \psi(d_p - x), & \text{if strains in steel are not limited,} \\ \varepsilon_{pm} + \psi(d_p - x) \le \varepsilon_{ud}, & \text{if strains in steel are limited,} \end{cases} \quad (5)$$

where d_s and d_p are the distances from the extreme compression fibre to the centroids of the reinforcing and prestressing steels respectively.

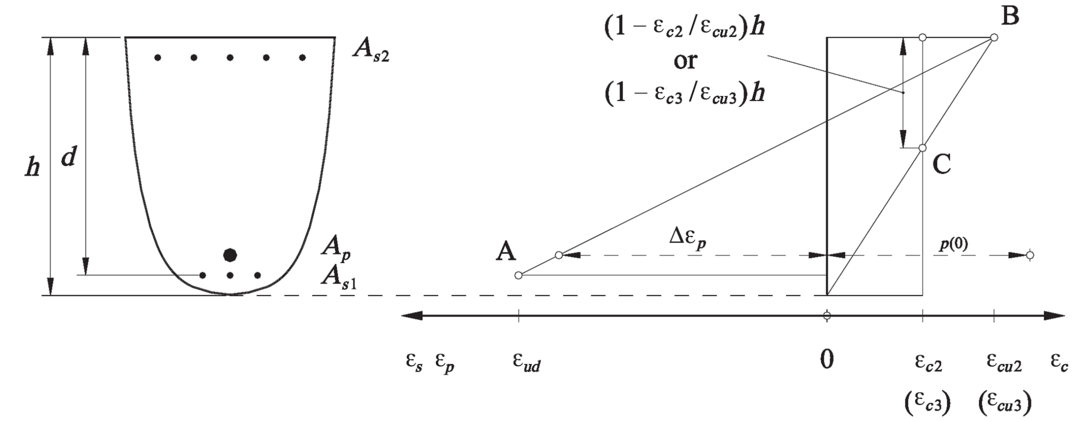

Fig 1. Possible strain distributions in the ultimate limit state [7]

3. Stress distribution of cross-section

For the concrete in compression a rectangular stress distribution is used. For the steel the stress-strain diagram with a horizontal or an inclined top branch is used [7]. To these diagrams (Fig 2) are added formulars to find stresses in region of strain hardening (inclined top branch). Taking into account Eqs (4) and (5), the stresses become:

diagram a, the stress in the reinforcing steel σ_s:

$$\sigma_s = \begin{cases} \psi(d_s - x)E_s & \text{, if } |\varepsilon_s| \le |\varepsilon_{yd}|, \text{ lower branch} \\ f_{yd} & \text{, if } |\varepsilon_s| \ge |\varepsilon_{yd}|, \text{ horizontal top branch} \\ f_{yd}\left(1 - \frac{\overline{E}_s}{E_s}\right) + \psi(d_s - x)\overline{E}_s, & \text{if } |\varepsilon_{yd}| \le |\varepsilon_s| \le |\varepsilon_{ud}|, \text{ inclined top branch;} \end{cases} \quad (6)$$

diagram b, the stress in the prestressing steel σ_p with the prestress σ_{pm}:

$$\sigma_p = \begin{cases} \sigma_{pm} + \psi(d_p - x)E_p, & \text{if } \varepsilon_p \le \varepsilon_{fpd}, \text{ lower branch} \\ f_{pd}, & \text{if } \varepsilon_p \ge \varepsilon_{fpd}, \text{ horizontal top branch} \\ f_{pd} - (f_{pd} - \sigma_{pm})\frac{\overline{E}_p}{E_p} + \psi(d_p - x)\overline{E}_p, & \text{if } \varepsilon_{fpd} \le \varepsilon_p \le \varepsilon_{ud}, \text{ inclined top branch}, \end{cases} \quad (7)$$

where the slopes of top branch of diagram for reinforcing and prestressing steel are $\overline{E}_s = \frac{f_{yd}(k-1)}{\varepsilon_{uk} - \varepsilon_{yd}}$ and $\overline{E}_p = \frac{f_{pd}(k-1)}{\varepsilon_{uk} - \varepsilon_{fpd}}$ respectively, E_s and E_p are the moduli of elasticity for reinforcing and prestressing steel. For the slope equations the following characteristic values are used:
f_{yd} and f_{pd} – design strength of reinforcing and prestressing steel,
ε_{yd} and ε_{fpd} – strains according to the design strength of reinforcing and prestressing steel,
ε_{uk} – characteristic strain of steel at maximum force.

4. Height of compression zone

4.1. Steel stress-strain diagram with horizontal top branch

We consider a symmetrical cross-section reinforced by a number of layers of steel and subjected to a design bending moment M_{Ed} and a design axial force N_{Ed} (Fig 3).

Summing axial forces gives the following equation of equilibrium:

$$\eta f_{cd} A_c - \sum_{i=1}^{n} \sigma_{si} A_{si} - \sum_{j=1}^{m} \sigma_{pj} A_{pj} - N_{Ed} = 0 \ , \quad (8)$$

where f_{cd} is the design compressive strength of concrete, η – a factor depending on the strength class for concrete, A_c – the effective area of concrete in compression, A_{si} and A_{pj} – the areas of reinforcing and prestressing steel respectively. Summations in Eq (8) are to be performed from $i = 1$ to *n* and $j = 1$ to *m*, where *n* and *m* are the numbers of reinforcing and prestressing steel layers respectively. The area A_c can be written as the sum of two parts:

$$A_c = \begin{cases} b^{*} \cdot x + \Delta A_c, & \text{if } \cdot x < h, \\ bh + \Delta A_c, & \text{if } \cdot x \ge h, \end{cases} \quad (9)$$

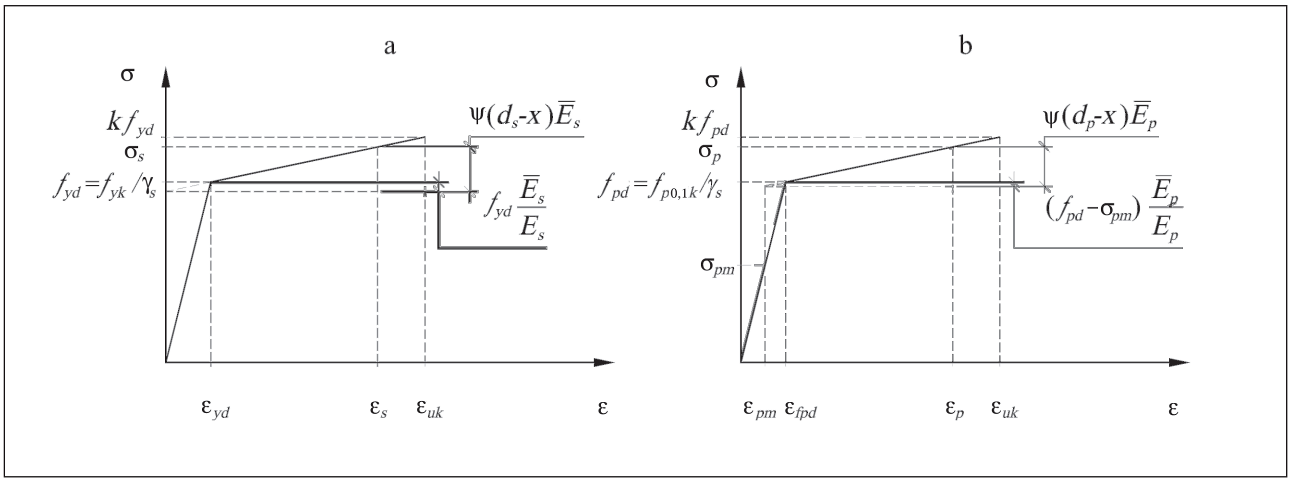

Fig 2. Design stress-strain diagrams for reinforcing steel (a) and prestressing steel (b)

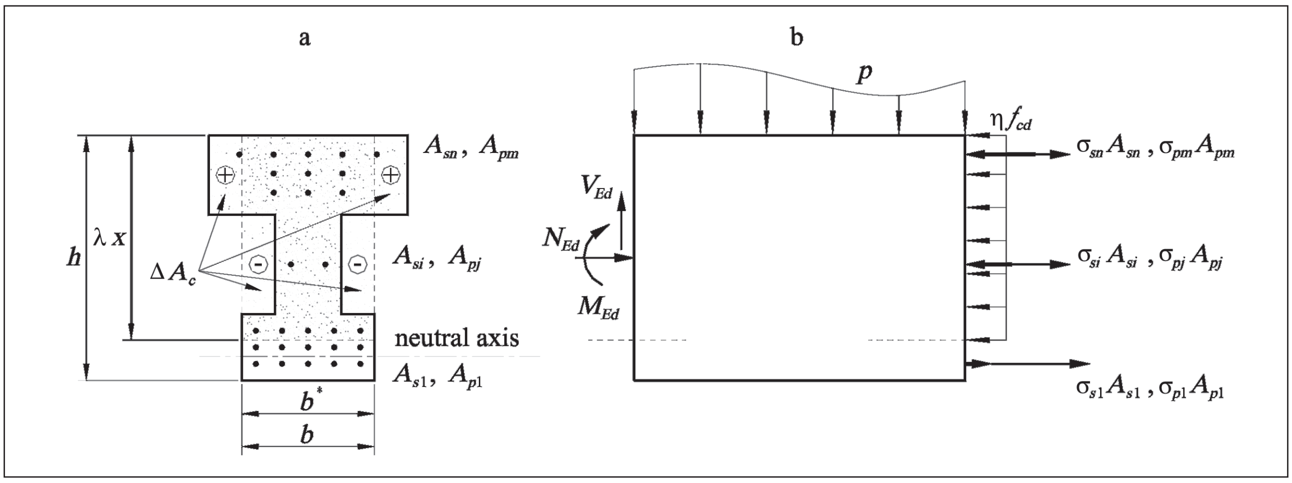

Fig 3. Cross-section (a); stresses and forces (b)

where b^* is the section width at the effective height of compression zone $\dot{e}x$, $\dot{e}$ is a factor depending on the strength class for concrete. The area ΔA_c is obtained by summing the "positive areas" and the "negative areas" (Fig 3) and can have various values depending on the shape of cross-section and the position of neutral axis.

Flexural strength calculations in the case $\dot{e}x \geq h$ can usually be carried out without determining the height of compression zone. Here we consider the case $\dot{e}x < h$.

Substituting Eq (6) and (7) into Eq (8) we get:

$$\eta f_{cd} b^* \dot{e}x + \eta f_{cd} \Delta A_c - \sum_{i=1}^{n} \psi(d_{si} - x) E_s A_{si} - \sum_{i=1}^{n} f_{yd} A_{si} -$$

$$\sum_{j=1}^{m} \left[\sigma_{pjm} + \psi(d_{pj} - x) E_p\right] A_{pj} - \sum_{j=1}^{m} f_{pd} A_{pj} - N_{Ed} = 0, \quad (10)$$

where members containing steel stresses define the sum of forces in steel layers:

$$\sum_{i=1}^{n} \psi(d_{si} - x) E_s A_{si}, \text{ if } |\sigma_s| < |f_{yd}|,$$

$$\sum_{i=1}^{n} f_{yd} A_{si}, \text{ if } |\sigma_s| = |f_{yd}|,$$

$$\sum_{j=1}^{m} \left[\sigma_{pjm} + \psi(d_{pj} - x) E_p\right] A_{pj}, \text{ if } \sigma_p < f_{pd},$$

$$\sum_{j=1}^{m} f_{pd} A_{pj}, \text{ if } \sigma_p = f_{pd}.$$

Next we multiply Eq (10) with distances between neutral axis and fibres with strains ε_{cu2}, ε_{c2} and consider Eq (3). To represent the result by one equation we use Macaulay function:

$$F_0(x) = \langle x - a \rangle^0 = \begin{cases} 0, \text{ if } x \leq a \\ 1, \text{ if } x > a. \end{cases} \quad (11)$$

Arranging the members in the equilibrium equation according to the power of height x, we get:

$$\eta f_{cd} b^* \dot{e} x^2 - \left[-\sum_{i=1}^{n} \sigma_{sc,u} A_{si} + \sum_{i=1}^{n} f_{yd} A_{si} + \right.$$

$$\sum_{j=1}^{m} (\sigma_{pjm} - \sigma_{pc,u}) A_{pj} + \sum_{j=1}^{m} f_{pd} A_{pj} \langle h - x \rangle^0 -$$

$$\left. \eta f_{cd} \Delta A_c + N_{Ed} + \eta f_{cd} b^* \dot{e} x_c \langle x - h \rangle^0 \right] x -$$

$$\left[\sum_{i=1}^{n} \sigma_{sc,u} A_{si} d_{si} - \sum_{i=1}^{n} f_{yd} A_{si} x_c \langle x - h \rangle^0 + \right.$$

$$\sum_{j=1}^{m} \left(\sigma_{pc,u} d_{pj} - \sigma_{pjm} x_c \langle x - h \rangle^0 \right) A_{pj} +$$

$$\left. (\eta f_{cd} \Delta A_c - N_{Ed}) x_c \langle x - h \rangle^0 \right] = 0, \quad (12)$$

where the reinforcing and prestressing steel conditional stresses $\sigma_{sc,u}$ and $\sigma_{pc,u}$ corresponding to the limiting compressive strains in concrete ε_{cu2} and ε_{c2} are:

$$\sigma_{sc,u} = \begin{cases} \varepsilon_{cu2} E_s, \text{ if } x \leq h, \\ \varepsilon_{c2} E_s, \text{ if } x > h, \end{cases} \quad \sigma_{pc,u} = \begin{cases} \varepsilon_{cu2} E_p, \text{ if } x \leq h, \\ \varepsilon_{c2} E_p, \text{ if } x > h. \end{cases}$$

Eq (12) is a quadratic equation of the form:

$$Ax^2 - Bx - C = 0, \quad (13)$$

and the height of compression zone is given by the positive root:

$$x = \frac{B + \sqrt{B^2 + 4AC}}{2A}, \quad (14)$$

where the coefficients A, B and C are:

$$A = \eta f_{cd} b^{*} \lambda , \qquad (15)$$

$$B = -\sum_{i=1}^{n} \sigma_{sc,u} A_{si} + \sum_{i=1}^{n} f_{yd} A_{si} + \sum_{j=1}^{m} \left(\sigma_{pjm} - \sigma_{pc,u}\right) A_{pj} + \sum_{j=1}^{m} f_{pd} A_{pj} \langle h - x \rangle^{0} - \eta f_{cd} \Delta A_c + N_{Ed} + \eta f_{cd} b^{*} \lambda x_c \langle x - h \rangle^{0} , \qquad (16)$$

$$C = \sum_{i=1}^{n} \sigma_{sc,u} A_{si} d_{si} - \sum_{i=1}^{n} f_{yd} A_{si} x_c \langle x - h \rangle^{0} + \sum_{j=1}^{m} \left(\sigma_{pc,u} d_{pj} - \sigma_{pjm} x_c \langle x - h \rangle^{0}\right) A_{pj} + \left(\eta f_{cd} \Delta A_c - N_{Ed}\right) x_c \langle x - h \rangle^{0} . \qquad (17)$$

The members

$\sum_{i=1}^{n} \sigma_{sc,u} A_{si}$, $\sum_{i=1}^{n} \sigma_{sc,u} A_{si} d_{si}$, $\sum_{j=1}^{m} \left(\sigma_{pjm} - \sigma_{pc,u}\right) A_{pj}$, $\sum_{j=1}^{m} \left(\sigma_{pc,u} d_{pj} - \sigma_{pjm} x_c \langle x - h \rangle^{0}\right) A_{pj}$ will be found according to steel layers, where the stresses are less than the yield stress and the members $\sum_{i=1}^{n} f_{yd} A_{si}$, $\sum_{i=1}^{n} f_{yd} A_{si} x_c \langle x - h \rangle^{0}$, $\sum_{j=1}^{m} f_{pd} A_{pj} \langle h - x \rangle^{0}$ according to steel layers, where the stresses are equal to the yield stress.

If the steel stresses are equal to the yield stress, then Eq (8) gives a linear equation in x:

$$\eta f_{cd} b^{*} \dot{e} x + \eta f_{cd} \Delta A_c - \sum_{i=1}^{n} f_{yd} A_{si} - \sum_{j=1}^{m} f_{pd} A_{pj} - N_{Ed} = 0 . \qquad (18)$$

4.2. Steel stress-strain diagram with an inclined top branch

Using the stress-strain diagrams with a horizontal and an inclined top branch for compression and tension zone respectively, Eq (8) becomes:

$$\eta f_{cd} b^{*} \dot{e} x + \eta f_{cd} \Delta A_c - \sum_{i=1}^{n} \psi \left(d_{si} - x\right) E_s A_{si} - \sum_{i=1}^{n} f_{yd} A_{si} - \sum_{i=1}^{n} \left[f_{yd} \left(1 - \frac{\overline{E}_s}{E_s}\right) + \psi \left(d_{si} - x\right) \overline{E}_s \right] A_{si} - \sum_{j=1}^{m} \left[\sigma_{pjm} + \psi \left(d_{pj} - x\right) E_p \right] A_{pj} - \sum_{j=1}^{m} \left[f_{pd} - \left(f_{pd} - \sigma_{pjm}\right) \frac{\overline{E}_p}{E_p} + \psi \left(d_{pj} - x\right) \overline{E}_p \right] A_{pj} - N_{Ed} = 0 , \qquad (19)$$

where the members containing steel stresses express the sum of forces in steel layers:

$$\sum_{i=1}^{n} \psi \left(d_{si} - x\right) E_s A_{si} , \text{ if } \sigma_s < f_{yd} ,$$

$$\sum_{i=1}^{n} \left[f_{yd} \left(1 - \frac{\overline{E}_s}{E_s}\right) + \psi \left(d_{si} - x\right) \overline{E}_s \right] A_{si} , \text{ if } f_{yd} \le \sigma_s \le k f_{yd} ,$$

$$\sum_{j=1}^{m} \left[\sigma_{pjm} + \psi \left(d_{pj} - x\right) E_p \right] A_{pj} , \text{ if } \sigma_p < f_{pd} ,$$

$$\sum_{j=1}^{m} \left[f_{pd} - \left(f_{pd} - \sigma_{pjm}\right) \frac{\overline{E}_p}{E_p} + \psi \left(d_{pj} - x\right) \overline{E}_p \right] A_{pj} , \text{ if } f_{pd} \le \sigma_p \le k f_{pd} .$$

Let us see the case, where the strain diagram in Fig 1 goes through the point B, then $\psi = \varepsilon_{cu2} / x$ and $\frac{\varepsilon_{cu2}}{\varepsilon_{ud} + \varepsilon_{cu2}} d \le x \le h$. Using the represented train of thought, we obtain the coefficients of quadratic equation as follows:

$$A = \eta f_{cd} b^{*} \dot{e} , \qquad (15)$$

$$B = -\sum_{i=1}^{n} \sigma_{sc,u} A_{si} + \sum_{i=1}^{n} f_{yd} A_{si} + \sum_{i=1}^{n} \left[f_{yd} \left(1 - \frac{\overline{E}_s}{E_s}\right) - \overline{\sigma}_{sc,u} \right] A_{si} + \sum_{j=1}^{m} \left(\sigma_{pjm} - \sigma_{pc,u}\right) A_{pj} + \sum_{j=1}^{m} \left[f_{pd} - \left(f_{pd} - \sigma_{pjm}\right) \frac{\overline{E}_p}{E_p} - \overline{\sigma}_{pc,u} \right] A_{pj} - \eta f_{cd} \Delta A_c + N_{Ed} , \qquad (20)$$

$$C = \sum_{i=1}^{n} \sigma_{sc,u} A_{si} d_{si} + \sum_{i=1}^{n} \overline{\sigma}_{sc,u} A_{si} d_{si} + \sum_{j=1}^{m} \sigma_{pc,u} A_{pj} d_{pj} + \sum_{j=1}^{m} \overline{\sigma}_{pc,u} A_{pj} d_{pj} , \qquad (21)$$

where

$$\overline{\sigma}_{sc,u} = \begin{cases} \varepsilon_{cu2} \overline{E}_s , \text{ if } x \le h \\ \varepsilon_{c2} \overline{E}_s , \text{ if } x > h, \end{cases} \qquad \overline{\sigma}_{pc,u} = \begin{cases} \varepsilon_{cu2} \overline{E}_p , \text{ if } x \le h \\ \varepsilon_{c2} \overline{E}_p , \text{ if } x > h. \end{cases}$$

Coefficients of equation $Ax^2 - Bx - C = 0$

Type of steel	Steel stress in layer	A	B: Part 1: Dependence on steel stresses	B: Part 2: Dependence on concrete strength and axial force	C: Part 1: Dependence on steel stresses	C: Part 2: Dependence on concrete strength and axial force
1	2	3	4	5	6	7
Reinforcing steel	$\lvert\sigma_s\rvert < \lvert f_{yd}\rvert$	$\eta f_{cd} b^* \lambda$	$-\sigma_{sc,u} A_s$	$-\eta f_{cd}\Delta A_c + N_{Ed} + \eta f_{cd} b^* \dot{e} x_c \langle x-h\rangle^0$	$\sigma_{sc,u} A_s d_s$	$(\eta f_{cd}\Delta A_c - N_{Ed})\cdot x_c\langle x-h\rangle^0$
	$\lvert\sigma_s\rvert = \lvert f_{yd}\rvert$		$f_{yd} A_s$		$-f_{yd} A_s x_c \langle x-h\rangle^0$	
	$f_{yd} \le \sigma_s \le kf_{yd}$ [1)]		$\left[f_{yd}\left(1-\frac{\bar{E}_s}{E_s}\right)-\bar{\sigma}_{sc,u}\right]A_s$		$\bar{\sigma}_{sc,u} A_s d_s$	
Prestressing steel	$\sigma_p < f_{pd}$		$(\sigma_{pm} - \sigma_{pc,u})A_p$		$(\sigma_{pc,u} d_p - \sigma_{pm} x_c \langle x-h\rangle^0)A_p$	
	$\sigma_p = f_{pd}$		$f_{pd} A_p \langle h-x\rangle^0$		—	
	$f_{pd} \le \sigma_p \le kf_{pd}$ [2)]		$\left[f_{pd} - (f_{pd} - \sigma_{pm})\frac{\bar{E}_p}{E_p} - \bar{\sigma}_{pc,u}\right]A_p$		$\bar{\sigma}_{pc,u} A_p d_p$	

[1)] reinforcing steel in tension zone

[2)] prestressing steel in tension zone

The height of compression zone is obtained from Eq (14).

The determination of coefficients *A*, *B* and *C* can also be carried out for both stress-strain diagrams by the Table given below. Coefficients *B* and *C* consist of 2 parts:

– part 1 represents the dependence on steel stresses and is calculable by summing up the members for each steel layer according to their stresses;
– part 2 is calculable depending on concrete strength and axial force.

The table can also be used to determine the height of compression zone when $\dot{e}x \ge h$. In that case coefficient $A = 0$ and contents of columns 5 and 7 will be turned into $(-\eta f_{cd} A_c + N_{Ed})$ and $(\eta f_{cd} A_c - N_{Ed})x_c$ respectively.

At the beginning of calculations the position of neutral axis shall be assumed. The following calculations will show, whether the assumption was right, or the calculations shall be repeated with another position of neutral axis. For control there are steel stresses (Eq (6), (7)), what cannot exceed the tensile strength and the stresses determined by strain distribution of section.

There are the following possibilities to find the coefficients of Eq (13) for the assumed position of neutral axis:

– to compare the steel stresses with the yield strength (or to compare respective strains);
– to compare the relative height of compression zone with the respective relative height to the yield strength.

5. Flexural strength calculation

Once the value of height of compression zone is known, the stresses in each layer of steel can be computed. In the reinforcing steel:

$$\sigma_s = \begin{cases} \sigma_{sc,u}\left(\frac{d_s}{x}-1\right), & \text{if } |\sigma_s| \le |f_{yd}| \text{ and } x \le h, \\ \sigma_{sc,u}\left(\frac{d_s - x}{x - x_c}\right), & \text{if } |\sigma_s| \le |f_{yd}| \text{ and } x > h, \\ f_{yd}, & \text{if } |\sigma_s| \ge |f_{yd}|, \\ f_{yd}\left(1-\frac{\bar{E}_s}{E_s}\right) + \bar{\sigma}_{sc,u}\left(\frac{d_s}{x}-1\right), & \text{if } f_{yd} \le \sigma_s \le kf_{yd}; \end{cases} \quad (22)$$

in the prestressing steel:

$$\sigma_p = \begin{cases} \sigma_{pm} + \sigma_{pc,u}\left(\dfrac{d_p}{x} - 1\right), \text{ if } \sigma_p \le f_{pd} \text{ and } x \le h, \\ \sigma_{pm} + \sigma_{pc,u}\left(\dfrac{d_p - x}{x - x_c}\right), \text{ if } \sigma_p \le f_{pd} \text{ and } x > h, \\ f_{pd}, \quad \text{if } \sigma_p \ge f_{pd}, \\ f_{pd} - \left(f_{pd} - \sigma_{pm}\right)\dfrac{\overline{E}_p}{E_p} + \overline{\sigma}_{pc,u}\left(\dfrac{d_p}{x} - 1\right), \\ \quad \text{if } f_{pd} \le \sigma_p \le k f_{pd}. \end{cases}$$

(23)

Finally the ultimate moment resistance of the section M_{Rd} can be found. For instance, taking moments about the mid-height of the section we obtain:

$$M_{Rd} = \eta f_{cd} b^* \dot{e} x(0{,}5h - 0{,}5\dot{e}x) + \eta f_{cd}\Delta S_c - \sum_{i=1}^{n}\sigma_{si} A_{si}(0{,}5h - d_{si}) - \sum_{j=1}^{m}\sigma_{pj} A_{pj}(0{,}5h - d_{pj}), \quad (24)$$

where ΔS_c is the first moment of the area ΔA_c about the mid-height of the section.

6. Example

The ultimate moment resistance of the section shown in Fig 4 is calculated by the proposed method.

Given: Concrete C35/45, $f_{cd} = 19{,}83$ N/mm², $\varepsilon_{cu2} = 0{,}0035$, $\eta = 1$, $\dot{e} = 0{,}8$.

Prestressing steel St1570/1770, $f_{pk} = 1770$ N/mm²,

$f_{pk} / \gamma_s = 1770/1{,}15 = 1539$ N/mm²,

$f_{p0,1k} = 1500$ N/mm²,

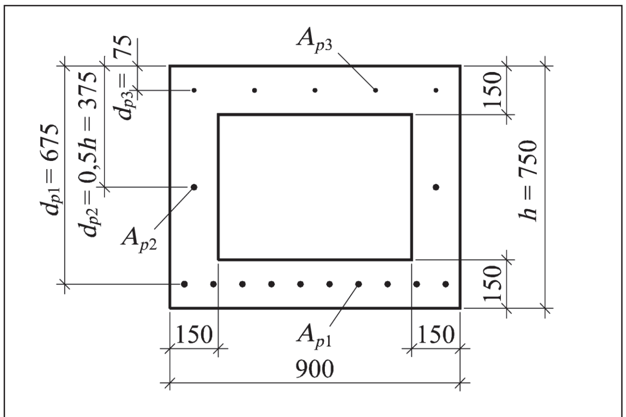

Fig 4. Boxed section

$f_{pd} = f_{p0,1k} / \gamma_s = 1500/1{,}15 = 1304$ N/mm²,

$k = f_{pk} / f_{p0,1k} = 1\,770/1\,500 = 1{,}180$, $\varepsilon_{uk} = 0{,}035$,

$E_p = 195\,000$ N/mm²,

$\varepsilon_{fpd} = f_{pd} / E_p = 1\,304/195\,000 = 0{,}006\,687$,

$$\overline{E}_p = \frac{f_{pd}(k-1)}{\varepsilon_{uk} - \varepsilon_{fpd}} = \frac{1\,304(1{,}180-1)}{0{,}035 - 0{,}006\,687} = 8\,290 \text{ N/mm²},$$

$\sigma_{pc,u} = \varepsilon_{cu2} E_p = 0{,}003\,5 \cdot 195\,000 = 682{,}5$ N/mm²,

$\overline{\sigma}_{pc,u} = \varepsilon_{cu2} \overline{E}_p = 0{,}003\,5 \cdot 8\,290 = 29{,}01$ N/mm²,

$A_{p1} = 2\,000$ mm² $(10\varnothing 18{,}0)$,

$A_{p2} = 400$ mm² $(2\varnothing 18{,}0)$, $A_{p3} = 260$ mm² $(5\varnothing 9{,}3)$,

$\sigma_{p1m} = \sigma_{p2m} = \sigma_{p3m} = 1\,000$ N/mm².

1) The calculations based on the stress-strain diagram with horizontal top branch.

Assuming that the steel in the layer 1 of strands yields and the compression zone extends into the web the coefficients of Eq (13) are:

$$A = \eta f_{cd} b^* \dot{e} = 1 \cdot 19{,}83 \cdot 300 \cdot 0{,}8 = 4759 \text{ N/mm},$$

$$B = f_{pd} A_{p1} + \sum_{j=2}^{3}(\sigma_{pjm} - \sigma_{pc,u})A_{pj} - \eta f_{cd}\Delta A_c =$$

$$1\,304 \cdot 2\,000 + (1\,000 - 682{,}5)400 + (1\,000 - 682{,}5)260 - 1 \cdot 19{,}83 \cdot 90\,000 = 1\,033 \cdot 10^3 \text{ N},$$

$$C = \sum_{j=2}^{3}\sigma_{pc,u} A_{pj} d_{pj} = 682{,}5 \cdot 400 \cdot 375 + 682{,}5 \cdot 260 \cdot 75 = 115{,}7 \cdot 10^6 \text{ Nmm},$$

where $\Delta A_c = 600 \cdot 150 = 90\,000 \text{ mm}^2$.

The height of compression zone:

$$x = \frac{B + \sqrt{B^2 + 4AC}}{2A} = \frac{1\,033 \cdot 10^3 + \sqrt{(1\,033 \cdot 10^3)^2 + 4 \cdot 4\,759 \cdot 115{,}7 \cdot 10^6}}{2 \cdot 4\,759} = 298 \text{ mm}.$$

The steel stresses:

$$\sigma_{p1} = \sigma_{p1m} + \sigma_{pc,u}\left(\frac{d_{p1}}{x} - 1\right) = 1\,000 + 682{,}5\left(\frac{675}{298} - 1\right) = 1861 \text{ N/mm²} > f_{pd} = 1\,304 \text{ N/mm²},$$

$$\sigma_{p2} = \sigma_{p2m} + \sigma_{pc,u}\left(\frac{d_{p2}}{x} - 1\right) = 1\,000 + 682{,}5\left(\frac{375}{298} - 1\right) =$$

$$1\ 175 \text{ N/mm}^2 < f_{pd} = 1\ 304 \text{ N/mm}^2,$$

$$\sigma_{p3} = \sigma_{p3m} + \sigma_{pc,u}\left(\frac{d_{p3}}{x} - 1\right) = 1\,000 + 682{,}5\left(\frac{75}{298} - 1\right) =$$

$$489{,}0 \text{ N/mm}^2,$$

thus the steel in the layer 1 yields $\sigma_{p1} = f_{pd}$. Taking moments about the mid-height of the section, we get the ultimate moment resistance of section:

$$M_{Rd} = \eta f_{cd} b^* \dot{e} x(0{,}5h - 0{,}5\dot{e}x) + \eta f_{cd} \Delta S_c -$$

$$\sum_{j=1}^{3} \sigma_{pj} A_{pj}(0{,}5h - d_{pj}) =$$

$$1 \cdot 19{,}83 \cdot 300 \cdot 0{,}8 \cdot 298(375 - 0{,}5 \cdot 0{,}8 \cdot 298) +$$

$$1 \cdot 19{,}83 \cdot 27{,}0 \cdot 10^6 - 1\,304 \cdot 2\,000(375 - 675) -$$

$$489{,}0 \cdot 260(375 - 75) = 1\,640 \cdot 10^6 \text{ Nmm},$$

where the first moment of the area ΔA_c about the mid-height of the section:

$$\Delta S_c = 90\,000 \cdot 300 = 27{,}0 \cdot 10^6 \text{ mm}^3 .$$

2) The calculations based on the stress-strain diagram with inclined top branch.

The coefficients of Eq (13) are:

$$B = \left[f_{pd} - (f_{pd} - \sigma_{p1m})\frac{\bar{E}_p}{E_p} - \bar{\sigma}_{pc,u} \right] A_{p1} +$$

$$\sum_{j=2}^{3} (\sigma_{pjm} - \sigma_{pc,u}) A_{pj} - \eta f_{cd} \Delta A_c =$$

$$\left[1\ 304 - (1\ 304 - 1\ 000)\frac{8\,290}{195\ 000} - 29{,}01\right] 2\,000 +$$

$$(1\ 000 - 682{,}5)400 + (1\ 000 - 682{,}5)260 -$$

$$1 \cdot 19{,}83 \cdot 90\ 000 = 949{,}0 \cdot 10^3 \text{ N},$$

$$C = \bar{\sigma}_{pc,u} A_{p1} d_{p1} + \sum_{j=2}^{3} \sigma_{pc,u} A_{pj} d_{pj} =$$

$$29{,}01 \cdot 2\,000 \cdot 675 + 682{,}5 \cdot 400 \cdot 375 +$$

$$682{,}5 \cdot 260 \cdot 75 = 154{,}8 \cdot 10^6 \text{ Nmm}.$$

The height of compression zone:

$$x = \frac{949{,}0 \cdot 10^3 + \sqrt{(949{,}0 \cdot 10^3)^2 + 4 \cdot 4\,759 \cdot 154{,}8 \cdot 10^6}}{2 \cdot 4\,759} =$$

$$306 \text{ mm}.$$

The steel stresses:

$$\sigma_{p1} = f_{pd} - (f_{pd} - \sigma_{p1m})\frac{\bar{E}_p}{E_p} + \bar{\sigma}_{pc,u}\left(\frac{d_{p1}}{x} - 1\right) =$$

$$1\ 304 - (1\ 304 - 1\ 000)\frac{8\ 290}{195\ 000} + 29{,}01\left(\frac{675}{306} - 1\right) =$$

$$1\ 326 \text{ N/mm}^2 < \frac{f_{pk}}{\gamma_s} = 1\ 539 \text{ N/mm}^2,$$

$$\sigma_{p2} = \sigma_{p2m} + \sigma_{pc,u}\left(\frac{d_{p2}}{x} - 1\right) = 1\ 000 + 682{,}5\left(\frac{375}{306} - 1\right) =$$

$$1\ 154 \text{ N/mm}^2 < f_{pd} = 1\ 304 \text{ N/mm}^2,$$

$$\sigma_{p3} = \sigma_{p3m} + \sigma_{pc,u}\left(\frac{d_{p3}}{x} - 1\right) = 1\ 000 + 682{,}5\left(\frac{75}{306} - 1\right) =$$

$$484{,}9 \text{ N/mm}^2.$$

The ultimate moment resistance of section:

$$M_{Rd} = 1 \cdot 19{,}83 \cdot 300 \cdot 0{,}8 \cdot 306(375 - 0{,}5 \cdot 0{,}8 \cdot 306) +$$

$$1 \cdot 19{,}83 \cdot 27{,}0 \cdot 10^6 - 1\,326 \cdot 2\,000(375 - 675) -$$

$$484{,}9 \cdot 260(375 - 75) = 1\,660 \cdot 10^6 \text{ Nmm}.$$

7. Conclusions

The flexural strength analysis of reinforced and prestressed concrete members is performed. A new practical method for calculating the height of the compression zone at ultimate moment resistance is presented. The method is applicable to symmetrical cross-sections with any number of the reinforcing and prestressing steel layers and is valid for both normal and high-strength concretes. The stresses in the concrete are derived from the rectangular stress distribution and the stresses in the steel from stress-strain diagrams with a horizontal or an inclined top branch. The proposed method is illustrated by a numerical example.

The essential points of the present method compared with other practical methods are:

– there is no need to concentrate reinforcement on one level;
– prestressing steel can be with different levels of prestressing;
– the method is valid for high-strength concrete, too.

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PAPRASTŲ IR IŠ ANKSTO ĮTEMPTŲ GELŽBETONINIŲ ELEMENTŲ ATSPARUMO LENKIMUI SKAIČIAVIMO METODAS

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Santrauka

Straipsnyje nagrinėjamas paprastų ir iš anksto įtemptų gelžbetoninių elementų atsparumas lenkimui, kai skerspjūvis vienodai apkrautas simetrijos ašies atžvilgiu. Pasiūlytas naujas gniuždomosios zonos aukščio skaičiavimo metodas. Jis tinka paprastiems ir didelio stiprumo betonams, kai įtempių pasiskirstymas skerspjūvyje įvairus. Pasiūlytasis metodas remiasi *Eurocode 2* išdėstytomis prielaidomis, supaprastinimais ir medžiagų charakteristikomis. Pasiūlytos skaičiavimo formulės stačiakampiam įtempių pasiskirstymui nustatyti gniuždomoje betono zonoje ir plieno tempimo įtempių diagramoms braižyti. Pateiktas praktinis metodo taikymo pavyzdys.

Reikšminiai žodžiai: skaičiavimo metodas, atsparumas lenkimui, paprastas ir iš anksto įtemptas gelžbetonis, betonas, simetriškas skerspjūvis, išlinkimo ašis, gniuždomosios zonos aukštis.

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