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Positive Solutions of nth-Order Boundary Value Problems with Integral Boundary Conditions

Ilkay Yaslan Karaca[a] **and Fatma Tokmak Fen**[b]

[a] *Department of Mathematics, Ege University*
35100 Bornova, Izmir, Turkey
[b] *Department of Mathematics, Gazi University*
06500 Teknikokullar, Ankara, Turkey
E-mail(*corresp.*): ilkay.karaca@ege.edu.tr
E-mail: fatmatokmak@gazi.edu.tr

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Abstract. In this paper, by using double fixed point theorem and a new fixed point theorem, some sufficient conditions for the existence of at least two and at least three positive solutions of an nth-order boundary value problem with integral boundary conditions are established, respectively. We also give two examples to illustrate our main results.

Keywords: boundary value problems, fixed point theorem, positive solutions, integral boundary conditions.

AMS Subject Classification: 34B15; 34B18.

1 Introduction

The theory of boundary value problems with integral boundary conditions for differential equations arises in different areas of applied mathematics and physics. Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and nonlocal boundary value problems as special cases. The existence and multiplicity of positive solutions for boundary value problems with integral boundary conditions have received a great deal of attentions. To identify a few, we refer the reader to [1, 2, 6, 7, 8, 9, 11, 14, 15] and references therein. On the other hand, there is not much studies of higher-order differential equations with integral boundary conditions in the literature, see [3, 4, 10, 12, 16].

In [6], by using Krasnoselskii's fixed point theorem, Boucherif investigated the existence of positive solutions of following nonlocal second-order boundary value problems with integral boundary conditions

$$\begin{cases} y'' = f\big(t, y(t)\big), \quad 0 < t < 1, \\ y(0) - ay'(0) = \displaystyle\int_0^1 g_0(s)y(s)ds, \\ y(1) - by'(1) = \displaystyle\int_0^1 g_1(s)y(s)ds. \end{cases}$$

In [3], Ahmad and Ntouyas considered the following n-th order differential inclusion with four-point integral boundary conditions

$$\begin{cases} x^{(n)}(t) \in F\big(t, x(t)\big), \quad 0 < t < 1, \\ x(0) = \alpha \displaystyle\int_0^\xi x(s)ds, \quad x'(0) = 0, \ x''(0) = 0, \ \ldots, \ x^{(n-2)}(0) = 0, \\ x(1) = \beta \displaystyle\int_\eta^1 x(s)ds, \quad 0 < \xi < \eta < 1. \end{cases}$$

The existence results were obtained by applying the nonlinear alternative of Leray–Schauder type and some suitable theorems of fixed point theory.

In [4], Ahmad and Ntouyas developed some existence results for the following nth-order boundary value problem with four-point nonlocal integral boundary conditions by using Krasnoselskii's fixed point theorem and Leray–Schauder degree theory

$$\begin{cases} x^{(n)}(t) = f\big(t, x(t)\big), \quad 0 < t < 1, \\ x(0) = \alpha \displaystyle\int_0^\xi x(s)ds, \quad x'(0) = 0, \ x''(0) = 0, \ \ldots, \ x^{(n-2)}(0) = 0, \\ x(1) = \beta \displaystyle\int_\eta^1 x(s)ds, \quad 0 < \xi < \eta < 1. \end{cases}$$

In [16], Zhang et al. investigated the existence of positive solutions of the following nth-order boundary value problem with integral boundary conditions by using the fixed point theory for strict set contraction operator

$$\begin{cases} x^{(n)}(t) + f\big(t, x(t), x'(t), x''(t), ..., x^{(n-2)}(t)\big) = \theta, \quad t \in J, \ t \neq t_k, \\ \Delta x^{(n-2)}\big|_{t=t_k} = -I_k\big(x^{(n-2)}(t_k)\big), \quad k = 1, 2, ..., m, \\ x^{(i)}(0) = \theta, \quad i = 0, 1, ..., n-3, \\ x^{(n-2)}(0) = x^{(n-2)}(1) = \displaystyle\int_0^1 g(t)x^{(n-2)}(t)dt. \end{cases}$$

In [10], Feng et al. studied the existence, nonexistence, and multiplicity of positive solutions for the following nth-order impulsive differential equations with integral boundary conditions

$$\begin{cases} x^{(n)}(t) + f\big(t, x(t)\big) = 0, \quad t \in J, \ t \neq t_k, \\ -\Delta x^{(n-1)}\big|_{t=t_k} = I_k\big(x(t_k)\big), \quad k = 1, 2, ..., m, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \displaystyle\int_0^1 h(t)x(t)dt. \end{cases}$$

Motivated by the above results, in this study, we aim to develop some existence results for the following nth-order boundary value problem (BVP) with integral boundary conditions

$$\begin{cases} u^{(n)}(t) + f\big(t, u(t)\big) = 0, \quad t \in [0,1], \\ au^{(n-2)}(0) - bu^{(n-1)}(0) = \displaystyle\int_0^1 g_1(s)u^{(n-2)}(s)ds, \\ cu^{(n-2)}(1) + du^{(n-1)}(1) = \displaystyle\int_0^1 g_2(s)u^{(n-2)}(s)ds, \\ u^{(i)}(0) = 0, \quad 0 \le i \le n-3, \end{cases} \tag{1.1}$$

where $n \ge 3$.

Throughout this paper we assume that following conditions hold:

$(C1)$ $a, b, c, d \in [0, +\infty)$ with $ac + ad + bc > 0$;

$(C2)$ $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$, where $\mathbb{R}_+ = [0, +\infty)$;

$(C3)$ $g_1, g_2 \in C([0,1], \mathbb{R}_+)$.

By using the double fixed point theorem [5], we get the existence of at least two positive solution for the BVP (1.1).

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to demonstrate our main results.

2 Preliminaries

In this section, we present auxiliary lemmas which will be used later.

We shall reduce problem (1.1) to an integral equation in $C([0,1])$. To this goal, firstly by means of the transformation

$$u^{(n-2)}(t) = y(t), \tag{2.1}$$

we convert BVP (1.1) into

$$\begin{cases} u^{(n-2)}(t) = y(t), \quad t \in [0,1], \\ u^{(i)}(0) = 0, \quad i = 1, 2, ..., n-3, \end{cases} \tag{2.2}$$

and

$$\begin{cases} -y''(t) + f\big(t, u(t)\big) = 0, \quad t \in [0,1], \\ ay(0) - by'(0) = \displaystyle\int_0^1 g_1(s)u(s)ds, \\ cy(1) + dy'(1) = \displaystyle\int_0^1 g_2(s)u(s)ds. \end{cases} \tag{2.3}$$

Lemma 1. *If $y \in C(J)$, then BVP (2.2) has a unique solution u and u can be expressed in the form*

$$u(t) = \int_0^t \frac{(t-s)^{n-3}}{(n-3)!} y(s)ds. \tag{2.4}$$

Proof. The proof follows by routine calculations. □

Set

$$\triangle := \begin{vmatrix} -\int_0^1 g_1(s)\,(b+as)\,ds & \rho - \int_0^1 g_1(s)\,(d+c(1-s))\,ds \\ \rho - \int_0^1 g_2(s)\,(b+as)\,ds & -\int_0^1 g_2(s)\,(d+c(1-s))\,ds \end{vmatrix} \tag{2.5}$$

and

$$\rho := ad + ac + bc. \tag{2.6}$$

Lemma 2. *Let $(C1)$–$(C3)$ hold. Assume that*

$(C4)$ $\triangle \neq 0$.

If $y \in C[0,1]$ is a solution of the equation

$$y(t) = \int_0^1 G(t,s)\, f\big(s,u(s)\big)ds + A(f)(b+at) + B(f)\big(d+c(1-t)\big), \tag{2.7}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} (b+as)\,(d+c(1-t))\,, & s \le t, \\ (b+at)\,(d+c(1-s))\,, & t \le s, \end{cases} \tag{2.8}$$

$$A(f) = \frac{1}{\triangle}\begin{vmatrix} \int_0^1 g_1(s)H(s)ds & \rho - \int_0^1 g_1(s)(d+c(1-s))ds \\ \int_0^1 g_2(s)H(s)ds & -\int_0^1 g_2(s)(d+c(1-s))ds \end{vmatrix}, \tag{2.9}$$

$$B(f) = \frac{1}{\triangle}\begin{vmatrix} -\int_0^1 g_1(s)(b+as)ds & \int_0^1 g_1(s)H(s)ds \\ \rho - \int_0^1 g_2(s)(b+as)ds & \int_0^1 g_2(s)H(s)ds \end{vmatrix}, \tag{2.10}$$

$$H(s) = \int_0^1 G(s,r)\, f\big(r,u(r)\big)dr,$$

then y is a solution of the BVP (2.3).

Proof. Let y satisfies the integral equation (2.7), then we will show that y is a solution of the BVP (2.3). Since y satisfies equation (2.7), then we have

$$y(t) = \int_0^1 G(t,s)\, f\big(s,u(s)\big)ds + A(f)(b+at) + B(f)\big(d+c(1-t)\big),$$

i.e.,

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{\rho}(b+as)\big(d+c(1-t)\big)f\big(s,u(s)\big)ds \\
&\quad + \int_t^1 \frac{1}{\rho}(b+at)\big(d+c(1-s)\big)f\big(s,u(s)\big)ds \\
&\quad + A(f)(b+at) + B(f)\big(d+c(1-t)\big), \\
y'(t) &= -\int_0^t \frac{c}{\rho}(b+as)f\big(s,u(s)\big)ds \\
&\quad + \int_t^1 \frac{a}{\rho}\big(d+c(1-s)\big)f\big(s,u(s)\big)ds + A(f)a - B(f)c.
\end{aligned}$$

So that

$$\begin{aligned}
y''(t) &= \frac{1}{\rho}\Big(-c(b+at) - a\big(d+c(1-t)\big)\Big)f\big(t,u(t)\big) \\
&= \frac{1}{\rho}\big(-(ad+ac+bc)\big)f\big(t,u(t)\big) \\
&= -f\big(t,u(t)\big).
\end{aligned}$$

Since

$$\begin{aligned}
y(0) &= \int_0^1 \frac{b}{\rho}\big(d+c(1-s)\big)f\big(s,u(s)\big)ds + A(f)b + B(f)(d+c), \\
y'(0) &= \int_0^1 \frac{a}{\rho}\big(d+c(1-s)\big)f\big(s,u(s)\big)ds + A(f)a - B(f)c,
\end{aligned}$$

we have that

$$\begin{aligned}
ay(0) - by'(0) &= B(f)\,(ad+ac+bc) \\
&= \int_0^1 g_1(s)\bigg[\int_0^1 G\,(s,r)\,f\big(r,u(r)\big)dr \\
&\qquad + A(f)(b+as) + B(f)\big(d+c(1-s)\big)\bigg]ds. \qquad (2.11)
\end{aligned}$$

Since

$$\begin{aligned}
y(1) &= \int_0^1 \frac{d}{\rho}\big(b+a(s)\big)f\big(s,u(s)\big)ds + A(f)(b+a) + B(f)d, \\
y'(1) &= -\int_0^1 \frac{c}{\rho}\big(b+a(s)\big)f\big(s,u(s)\big)ds + A(f)a - B(f)c,
\end{aligned}$$

we have that

$$\begin{aligned}
cy(1) + dy'(1) &= A(f)\,(ad+ac+bc) \\
&= \int_0^1 g_2(s)\bigg[\int_0^1 G\,(s,r)\,f\big(r,u(r)\big)dr \\
&\qquad + A(f)(b+as) + B(f)\big(d+c(1-s)\big)\bigg]ds. \qquad (2.12)
\end{aligned}$$

From (2.6), (2.11) and (2.12), we get that

$$\begin{cases} \left[-\int_0^1 g_1(s)\,(b+as)\,ds\right] A(f) + \left[\rho - \int_0^1 g_1(s)\big(d+c(1-s)\big)ds\right] B(f) \\ \quad = \int_0^1 g_1(s) \left(\int_0^1 G\,(s,r)\, f\big(r,u(r)\big)dr\right) ds, \\ \left[\rho - \int_0^1 g_2(s)\,(b+as)\,ds\right] A(f) + \left[-\int_0^1 g_2(s)\big(d+c(1-s)\big)ds\right] B(f) \\ \quad = \int_0^1 g_2(s) \left(\int_0^1 G\,(s,r)\, f\big(r,u(r)\big)dr\right) ds, \end{cases},$$

which implies that $A(f)$ and $B(f)$ satisfy (2.9) and (2.10), respectively. □

Lemma 3. *Let* $(C1)$*–*$(C3)$ *hold. Assume*

$(C5)$ $\triangle < 0,\ \rho - \int_0^1 g_2(s)(b+as)ds > 0,\ a - \int_0^1 g_1(s)ds > 0.$

Then for $y \in C[0,1]$ *with* $f,\ q \geq 0$, *the solution* y *of the problem* (2.3) *satisfies* $y(t) \geq 0$ *for* $t \in [0,1]$.

Proof. It is an immediate subsequence of the facts that $G \geq 0$ on $[0,1] \times [0,1]$ and $A(f) \geq 0,\ B(f) \geq 0$. □

Lemma 4. *Let* $(C1)$*-*$(C3)$ *and* $(C5)$ *hold. Assume that*

$(C6)$ $c - \int_0^1 g_2(s)ds < 0.$

Then the solution $y \in C[0,1]$ *of the problem* (2.3) *satisfies* $y'(t) \geq 0$ *for* $t \in [0,1]$.

Proof. Assume that the inequality $y'(t) < 0$ holds. Since $y'(t)$ is nonincreasing on $[0,1]$, one can verify that

$$y'(1) \leq y'(t), \quad t \in [0,1].$$

From the boundary conditions of the problem (2.3), we have

$$-\frac{c}{d}y(1) + \frac{1}{d}\int_0^1 g_2(s)y(s)ds \leq y'(t) < 0.$$

The last inequality yields

$$-cy(1) + \int_0^1 g_2(s)y(s)ds < 0.$$

Therefore, we obtain that

$$\int_0^1 g_2(s)y(1)ds < \int_0^1 g_2(s)y(s)ds < cy(1),$$

i.e.,

$$\left(c - \int_0^1 g_2(s)ds\right) y(1) > 0.$$

According to Lemma 3, we have that $y(1) \geq 0$. So, $c - \int_0^1 g_2(s)ds > 0$. However, this contradicts to condition $(C6)$. Consequently, $y'(t) \geq 0$ for $t \in [0,1]$. $\square$

Let the Banach space $\mathbb{B} = \mathcal{C}([0,1])$ be equipped with the norm $\|y\| = \max_{t\in[0,1]} |y(t)|$, and we define a cone $\mathcal{P}$ in $\mathbb{B}$ by

$$\mathcal{P} = \big\{y \in \mathbb{B} : y(t) \text{ is nonnegative, nondecreasing and concave on } [0,1]\big\}. \tag{2.13}$$

Lemma 5. *Let $y \in \mathcal{P}$ and $0 < \eta < 1$. Then,*

$$\min_{t\in[\eta,1]} y(t) \geq \eta \|y\|.$$

Proof. Since $y \in \mathcal{P}$ we know that $y(t)$ is concave on $[0,1]$. So, $\min_{t\in[\eta,1]} y(t) = y(\eta)$ and $\|y\| = \max_{t\in[0,1]} y(t) = y(1)$. Since the graph of y is concave down on $[0,1]$, we have

$$\frac{y(1) - y(0)}{1} \leq \frac{y(\eta) - y(0)}{\eta},$$

i.e., $y(\eta) \geq \eta y(1) + (1-\eta)y(0)$. So, $y(\eta) \geq \eta y(1)$. The proof is complete. $\square$

We define the operator $T : \mathbb{B} \to \mathbb{B}$ by

$$(Ty)(t) = \int_0^1 G(t,s)\, F\big(s, y(s)\big) ds + A(f)(b + at) + B(f)\big(d + c(1-t)\big), \tag{2.14}$$

where $F(t, y(t)) = f(t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr)$, G, $A(f)$ and $B(f)$ are defined as in (2.8), (2.9) and (2.10), respectively.

Solving BVP (1.1) is equivalent to finding fixed points of the operator T defined by (2.14).

Lemma 6. *Let $(C1)$–$(C6)$ hold. Then $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. For all $y \in \mathcal{P}$, Lemmas 2, 3, 4 and the definition of T, we have

$$(Ty)(t) \geq 0, \qquad (Ty)'(t) \geq 0, \quad \text{and} \quad (Ty)'(t) \text{ is concave on } [0,1].$$

Then $Ty \in \mathcal{P}$. So T is an operator from $\mathcal{P}$ to $\mathcal{P}$. By Arzela-Ascoli theorem, we can easily prove that operator T is completely continuous. $\square$

3 Main Results

In this section, our objective is to establish the existence of at least two positive solutions and three positive solutions for the BVP (1.1) by using double fixed point theorem [5] and a new fixed point theorem [13], respectively.

For a nonnegative continuous functional γ on a cone $\mathcal{P}$ in a real Banach space $\mathbb{B}$, and each $d > 0$, we set

$$\mathcal{P}(\gamma, d) = \big\{y \in \mathcal{P} : \gamma(y) < d\big\}.$$

Lemma 7 [Double Fixed Point Theorem]. [5] *Let $\mathcal{P}$ be a cone in a real Banach space $\mathbb{B}$. Let α and γ be increasing, nonnegative, continuous functionals on $\mathcal{P}$, and let β be a nonnegative, continuous functional on $\mathcal{P}$ with $\beta(0) = 0$ such that, for some $l > 0$ and $M > 0$,*

$$\gamma(y) \le \beta(y) \le \alpha(y) \quad \textit{and} \quad \|y\| \le M\gamma(y)$$

for all $y \in \overline{\mathcal{P}(\gamma, l)}$. Suppose that there exist positive numbers j and k with $j < k < l$ such that

$$\beta(\lambda y) \le \lambda\beta(y), \quad \textit{for } 0 \le \lambda \le 1 \quad \textit{and} \quad y \in \partial\mathcal{P}(\beta, k)$$

and $T : \overline{\mathcal{P}(\gamma, l)} \to \mathcal{P}$ is a completely continuous operator such that:

(i) $\gamma(Ty) > l$ for all $y \in \partial\mathcal{P}(\gamma, l)$;

(ii) $\beta(Ty) < k$ for all $y \in \partial\mathcal{P}(\beta, k)$;

(iii) $\mathcal{P}(\alpha, j) \ne \emptyset$ and $\alpha(Ty) > j$ for all $x \in \partial\mathcal{P}(\alpha, j)$.

Then T has at least two fixed points, y_1 and y_2 belonging to $\overline{\mathcal{P}(\gamma, l)}$ such that

$$j < \alpha(y_1) \quad \textit{with } \beta(y_1) < k, \quad k < \beta(y_2) \quad \textit{with } \gamma(y_2) < l.$$

Lemma 8. [13] *Let $\mathcal{P}$ be a cone in a real Banach space $\mathbb{B}$. Let φ, θ and ψ be three increasing, nonnegative and continuous functionals on $\mathcal{P}$. There are constants $v > 0$ and $L > 0$ such that*

$$\psi(y) \le \theta(y) \le \varphi(y), \quad \|y\| \le \tilde{M}\psi(y)$$

for all $y \in \overline{\mathcal{P}(\psi, v)}$. Suppose there exists a completely continuous operator $T : \overline{\mathcal{P}(\psi, v)} \to \mathcal{P}$ and constants $0 < h < p < v$ such that

(i) $\psi(Ty) < v$ for all $y \in \partial\mathcal{P}(\psi, v)$;

(ii) $\theta(Ty) > p$ for all $y \in \partial\mathcal{P}(\theta, p)$;

(iii) $\mathcal{P}(\varphi, h) \ne \emptyset$ and $\varphi(Ty) < h$ for all $y \in \partial\mathcal{P}(\varphi, h)$.

Then T has at least three fixed points, y_1, y_2 and $y_3 \in \overline{\mathcal{P}(\psi, v)}$ such that

$$0 \le \varphi(y_1) < h < \varphi(y_2), \qquad \theta(y_2) < p < \theta(y_3), \qquad \psi(y_3) < v.$$

Now we consider the existence of at least two positive solutions for the nth-order boundary value problem (1.1) by using double fixed point theorem [5].

Let $0 < \eta < \xi < 1$ and define the increasing, nonnegative, continuous functionals α, γ, and nonnegative, continuous functional β on $\mathcal{P}$ by

$$\alpha(y) = \max_{t\in[0,\xi]} y(t) = y(\xi), \quad \beta(y) = \max_{t\in[0,\eta]} y(t) = y(\eta),$$
$$\gamma(y) = \min_{t\in[\eta,1]} y(t) = y(\eta).$$

It is obvious that for each $y \in \mathcal{P}$,

$$\gamma(y) = \beta(y) \leq \alpha(y).$$

In addition, from by Lemma 5, for each $y \in \mathcal{P}$,

$$\|y\| \leq \frac{1}{\eta} \min_{t\in[\eta,1]} y(t) = \frac{1}{\eta}\gamma(y).$$

Thus,

$$\|y\| \leq \frac{1}{\eta}\gamma(y), \quad \forall y \in \mathcal{P}.$$

For the convenience, we denote

$$A = \frac{1}{\triangle}\begin{vmatrix} \int_0^1 g_1(s)(\int_0^1 G(s,r)\,dr)ds & \rho - \int_0^1 g_1(s)(d + c(1-s))ds \\ \int_0^1 g_2(s)(\int_0^1 G(s,r)\,dr)ds & -\int_0^1 g_2(s)(d + c(1-s))ds \end{vmatrix},$$
$$B = \frac{1}{\triangle}\begin{vmatrix} -\int_0^1 g_1(s)(b+as)ds & \int_0^1 g_1(s)(\int_0^1 G(s,r)\,dr)ds \\ \rho - \int_0^1 g_2(s)(b+as)ds & \int_0^1 g_2(s)(\int_0^1 G(s,r)\,dr)ds \end{vmatrix},$$
$$L = \int_\xi^1 G(\eta,s)ds$$
$$N = \int_0^1 G(\eta,s)ds + A(b + a\eta) + B\big(d + c(1-\eta)\big).$$

Theorem 1. *Suppose that assumptions* $(C1)$–$(C6)$ *are satisfied. Let there exist positive numbers* $j < k < l$ *such that*

$$0 < j < \frac{L}{N}k < \frac{L\eta^2\,(\xi-\eta)^{n-2}}{N(n-2)!}l,$$

and assume that f *satisfies the following conditions*

$(C7)$ $f(t,u) > \frac{l}{L}$ *for all* $(t,u) \in [\xi,1] \times [\frac{\eta(\xi-\eta)^{n-2}}{(n-2)!}l, \frac{l}{\eta}]$,

$(C8)$ $f(t,u) < \frac{k}{N}$ *for all* $(t,u) \in [0,1] \times [0,\frac{k}{\eta}]$,

$(C9)$ $f(t,u) > \frac{j}{L}$ *for all* $(t,u) \in [\xi,1] \times [0,\frac{j}{\eta}]$.

Then the boundary value problem (1.1) *has at least two positive solutions.*

Proof. We define the completely continuous operator T by (2.14). So, it is easy to check that $T:\overline{\mathcal{P}(\gamma,l)}\to\mathcal{P}$. We now show that all the conditions of Lemma 7 are satisfied. In order to show that condition (i) of Lemma 7, we choose $y\in\partial\mathcal{P}(\gamma,l)$. Then $\gamma(y)=\min_{t\in[\eta,1]}y(t)=y(\eta)=l$, this implies that $l\le y(t)$ for $t\in[\eta,1]$. Recalling that $\|y\|\le\frac{1}{\eta}\gamma(y)=\frac{1}{\eta}l$, we get

$$l\le y(t)\le\frac{l}{\eta},\quad t\in[\eta,1].$$

It is clear that

$$\int_0^t\frac{(t-r)^{n-3}}{(n-3)!}y(r)dr\le y(t)\le\|y\|,\quad t\in[0,1]. \tag{3.1}$$

Since for $t\in[\eta,1]$ the following inequality holds

$$\begin{aligned}y(t)&\ge\int_0^t\frac{(t-r)^{n-3}}{(n-3)!}y(r)dr\ge\int_\eta^t\frac{(t-r)^{n-3}}{(n-3)!}y(r)dr\\&\ge\int_\eta^t\frac{(t-r)^{n-3}}{(n-3)!}\eta\|y\|\,dr,\end{aligned}$$

for $t\in[\xi,1]$ we have

$$\begin{aligned}\int_0^t\frac{(t-r)^{n-3}}{(n-3)!}y(r)dr&\ge\int_\eta^\xi\frac{(\xi-r)^{n-3}}{(n-3)!}\eta\|y\|\,dr\\&=\eta\|y\|\frac{(\xi-\eta)^{n-2}}{(n-2)!}\ge\eta y(t)\frac{(\xi-\eta)^{n-2}}{(n-2)!}.\end{aligned} \tag{3.2}$$

So, from (3.1) and (3.2) we get

$$\left(t,\int_0^t\frac{(t-r)^{n-3}}{(n-3)!}y(r)dr\right)\in[\xi,1]\times\left[\frac{\eta(\xi-\eta)^{n-2}}{(n-2)!}l,\frac{l}{\eta}\right].$$

Then assumption $(C7)$ implies

$$f(t,u)>\frac{l}{L},\quad\text{for all }(t,u)\in[\xi,1]\times\left[\frac{\eta(\xi-\eta)^{n-2}}{(n-2)!}l,\frac{l}{\eta}\right].$$

Therefore,

$$\begin{aligned}\gamma(Ty)&=\min_{t\in[\eta,1]}(Ty)(t)=(Ty)(\eta)\\&=\int_0^1G(\eta,s)F\big(s,y(s)\big)ds+A(f)(b+a\eta)+B(f)\big(d+c(1-\eta)\big)\\&\ge\int_0^1G(\eta,s)F\big(s,y(s)\big)ds\ge\int_\xi^1G(\eta,s)F\big(s,y(s)\big)ds\\&>\frac{l}{L}\int_\xi^1G(\eta,s)\,ds=l.\end{aligned}$$

Hence, condition (i) is satisfied.

Secondly, we show that (ii) of Lemma 7 is satisfied. For this, we select $y \in \partial \mathcal{P}(\beta, k)$. Then, $\beta(y) = \max_{t\in[0,\eta]} y(t) = y(\eta) = k$, this means $0 \le y(t) \le k$, for all $t \in [0, \eta]$. Noticing that $\|y\| \le \frac{1}{\eta}\gamma(y) = \frac{1}{\eta}\beta(y) = \frac{1}{\eta}k$, we get

$$0 \le y(t) \le \frac{k}{\eta} \quad \text{for } 0 \le t \le 1.$$

From (3.1), we have

$$\left(t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr\right) \in [0,1] \times \left[0, \frac{k}{\eta}\right].$$

Then, assumption $(C8)$ implies

$$f(t,u) < \frac{k}{N} \quad \text{for all } (t,u) \in [0,1] \times \left[0, \frac{k}{\eta}\right].$$

Therefore

$$\begin{aligned}
\beta(Ty) &= \max_{t\in[0,\eta]} (Ty)(t) = (Ty)(\eta) \\
&= \int_0^1 G(\eta, s)\, F\big(s, y(s)\big) ds + A(f)(b + a\eta) + B(f)\big(d + c(1-\eta)\big) \\
&< \frac{k}{N} \left(\int_0^1 G(\eta, s)\, ds + A(b + a\eta) + B\big(d + c(1-\eta)\big) \right) = k.
\end{aligned}$$

So, we get $\beta(Ty) < k$. Hence, condition (ii) is satisfied.

Finally, we show that the condition (iii) of Lemma 7 is satisfied. We note that $y(t) = \frac{j}{3}$, $0 \le t \le 1$ is a member of $\mathcal{P}(\alpha, j)$, and so $\mathcal{P}(\alpha, j) \ne \emptyset$.

Now, let $y \in \partial \mathcal{P}(\alpha, j)$. Then $\alpha(y) = \max_{t\in[0,\xi]} y(t) = y(\xi) = j$. This implies that $y(t) \ge j$ for $t \in [\xi, 1]$. Recalling that $\|y\| \le \frac{1}{\eta}\gamma(y) \le \frac{1}{\eta}\alpha(y) = \frac{j}{\eta}$, we get

$$j \le y(t) \le \frac{j}{\eta}, \quad t \in [\xi, 1].$$

From (3.1), we have

$$\left(t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr\right) \in [\xi,1] \times \left[0, \frac{j}{\eta}\right].$$

By assumption $(C9)$,

$$f(t,u) > \frac{j}{L} \quad \text{for all } (t,u) \in [\xi,1] \times \left[0, \frac{j}{\eta}\right].$$

Then,

$$\begin{aligned}
\alpha(Ty) &= \max_{t\in[0,\xi]} (Ty)(t) = (Ty)(\xi) \\
&= \int_0^1 G(\xi, s)\, F\big(s, y(s)\big) ds + A(f)(b + a\xi) + B(f)\big(d + c(1-\xi)\big)
\end{aligned}$$

$$\geq \int_0^1 G(\xi,s)\, F\big(s,y(s)\big)ds \geq \int_\xi^1 G(\xi,s)\, F\big(s,y(s)\big)ds$$
$$\geq \int_\xi^1 G(\eta,s)\, F\big(s,y(s)\big)ds > \frac{j}{L}\int_\xi^1 G(\eta,s)\, ds = j.$$

So, we get $\alpha(Ty) > j$. Thus, (iii) of Lemma 7 is satisfied.

Since all conditions of Lemma 7 are satisfied, T has at least two fixed points, y_1 and y_2 belonging to $\overline{\mathcal{P}(\gamma,l)}$ of BVP (2.3) such that

$$j < \alpha(y_1) \quad \text{with } \beta(y_1) < k, \quad k < \beta(y_2) \quad \text{with } \gamma(y_2) < l.$$

Then the nth-order BVP (1.1) has at least two positive solutions

$$u_i(t) = \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y_i(r)dr \quad (i = 1,2).$$

The proof is complete. □

Now we consider the existence of at least three positive solutions for the nth-order boundary value problem (1.1) by the fixed point theorem in [13].

Let $0 < \mu < \zeta < 1$ and define the increasing, nonnegative, continuous functionals ψ, θ, and φ on $\mathcal{P}$ by

$$\psi(y) = \max_{t\in[0,\mu]} y(t) = y(\mu), \quad \theta(y) = \min_{t\in[\mu,\zeta]} y(t) = y(\mu),$$
$$\varphi(y) = \max_{t\in[0,\zeta]} y(t) = y(\zeta).$$

It is obvious that for each $y \in \mathcal{P}$, $\psi(y) = \theta(y) \leq \varphi(y)$. In addition, for each $y \in \mathcal{P}$, since y is concave on $[0,1]$ we get $\psi(y) = y(\mu) \geq \mu y(1)$. Thus,

$$\|y\| \leq \frac{1}{\mu}\psi(y) \quad \forall y \in \mathcal{P}.$$

For the convenience, we denote

$$\Omega = \int_\zeta^1 G(\mu,s)ds, \quad \Lambda = \int_0^1 G(\zeta,s)ds + A(b+a\zeta) + B\big(d+c(1-\zeta)\big).$$

Theorem 2. *Suppose that assumptions $(C1)$–$(C6)$ are satisfied. Let there exist positive numbers $h < p < v$ such that*

$$\frac{h(n-2)!}{\mu^2\,(\zeta-\mu)^{n-2}} < p < \frac{\Omega}{\Lambda}v,$$

and assume that f satisfies the following conditions

$(C10)$ $f(t,u) < \frac{v}{\Lambda}$ *for all* $(t,u) \in [0,1]\times[0,\frac{v}{\mu}]$,

$(C11)$ $f(t,u) > \frac{p}{\Omega}$ *for all* $(t,u) \in [\zeta,1]\times[\frac{\mu(\zeta-\mu)^{n-2}}{(n-2)!}p, \frac{p}{\mu}]$, *(C12)* $f(t,u) < \frac{h}{\Lambda}$ *for all* $(t,u) \in [0,1] \times [0, \frac{h}{\mu}]$.

Then the boundary value problem (1.1) *has at least three positive solutions.*

Proof. We define the completely continuous operator T by (2.14). So, it is easy to check that $T : \overline{\mathcal{P}(\psi, v)} \to \mathcal{P}$. We now show that all the conditions of Lemma 8 are satisfied. In order to show that condition (i) of Lemma 8, we choose $y \in \partial\mathcal{P}(\psi, v)$. Then $\psi(y) = \max_{t\in[0,\mu]} y(t) = y(\mu) = v$, this means $0 \le y(t) \le v$, for all $t \in [0,\mu]$. Recalling that $\|y\| \le \frac{1}{\mu}\psi(y) = \frac{1}{\mu}v$, we get

$$0 \le y(t) \le \frac{v}{\mu} \quad \text{for } 0 \le t \le 1.$$

From (3.1), we have

$$\left(t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr\right) \in [0,1] \times \left[0, \frac{v}{\mu}\right].$$

Then, assumption $(C10)$ implies

$$f(t,u) < \frac{v}{\Lambda} \quad \text{for all } (t,u) \in [0,1] \times \left[0, \frac{v}{\mu}\right].$$

Therefore

$$\begin{aligned}
\psi(Ty) &= \max_{t\in[0,\mu]} (Ty)(t) = (Ty)(\mu) \\
&= \int_0^1 G(\mu, s)\, F\big(s, y(s)\big)ds + A(f)(b + a\mu) + B(f)\big(d + c(1-\mu)\big) \\
&< \frac{v}{\Lambda}\left(\int_0^1 G(\zeta, s)\, ds + A(b + a\zeta) + B\big(d + c(1-\zeta)\big)\right) \\
&= v.
\end{aligned}$$

So, we get $\psi(Ty) < v$. Hence, condition (i) of Lemma 8 is satisfied.

Secondly, we show that (ii) of Lemma 8 is satisfied. For this, we choose $y \in \partial\mathcal{P}(\theta, p)$. Then, $\theta(y) = \min_{t\in[\mu,\zeta]} y(t) = y(\mu) = p$, this means $y(t) \ge p$, for all $t \in [\mu, 1]$. Noticing that $\|y\| \le \frac{1}{\mu}\psi(y) \le \frac{1}{\mu}\theta(y) = \frac{p}{\mu}$, we get

$$p \le y(t) \le \frac{p}{\mu} \quad \text{for } t \in [\mu, 1].$$

Using (3.1), for $t \in [\mu, 1]$ the following inequality holds

$$\begin{aligned}
y(t) &\ge \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr \ge \int_\mu^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr \\
&\ge \int_\mu^t \frac{(t-r)^{n-3}}{(n-3)!} \mu \|y\|\, dr,
\end{aligned}$$

and for $t \in [\zeta, 1]$ we have from the above inequality

$$\int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr \geq \int_\mu^\zeta \frac{(\zeta-r)^{n-3}}{(n-3)!} \mu \|y\| dr \\ = \mu \|y\| \frac{(\zeta-\mu)^{n-2}}{(n-2)!} \geq \mu y(t) \frac{(\zeta-\mu)^{n-2}}{(n-2)!}. \tag{3.3}$$

So, from (3.1) and (3.3) we get

$$\left(t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr\right) \in [\zeta, 1] \times \left[\frac{\mu(\zeta-\mu)^{n-2}}{(n-2)!} p, \frac{p}{\mu}\right].$$

Then assumption $(C11)$ implies

$$f(t,u) > \frac{p}{\Omega} \quad \text{for all } (t,u) \in [\zeta, 1] \times \left[\frac{\mu(\zeta-\mu)^{n-2}}{(n-2)!} p, \frac{p}{\mu}\right].$$

Therefore,

$$\begin{aligned}\theta(Ty) &= \min_{t\in[\mu,\zeta]} (Ty)(t) = (Ty)(\mu) \\ &= \int_0^1 G(\mu,s) F\big(s,y(s)\big)ds + A(f)(b+a\mu) + B(f)\big(d+c(1-\mu)\big) \\ &\geq \int_0^1 G(\mu,s) F\big(s,y(s)\big)ds \geq \int_\zeta^1 G(\mu,s) F\big(s,y(s)\big)ds \\ &> \frac{p}{\Omega} \int_\zeta^1 G(\mu,s)\, ds = p.\end{aligned}$$

Hence, condition (ii) of Lemma 8 is satisfied.

Finally, we show that the condition (iii) of Lemma 8 is satisfied. We note that $y(t) = \frac{2h}{3}$, $t \in [0,1]$ is a member of $\mathcal{P}(\varphi, h)$, and so $\mathcal{P}(\varphi, h) \neq \emptyset$.

Now, let $y \in \partial\mathcal{P}(\varphi, h)$. Then $\varphi(y) = \max_{t\in[0,\zeta]} y(t) = y(\zeta) = h$. This implies $0 \leq y(t) \leq h$, $t \in [0,\zeta]$. Noticing that $\|y\| \leq \frac{1}{\mu}\psi(u) \leq \frac{1}{\mu}\varphi(y) = \frac{h}{\mu}$, we get

$$0 \leq y(t) \leq \frac{h}{\mu}, \quad \text{for } t \in [0,1].$$

From (3.1),we have

$$\left(t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r)dr\right) \in [0,1] \times \left[0, \frac{h}{\mu}\right].$$

By assumption $(C12)$, we have $f(t,u) < \frac{h}{\Lambda}$, for all $(t,u) \in [0,1] \times [0, \frac{h}{\eta}]$. Therefore, we obtain

$$\begin{aligned}\varphi(Ty) &= \max_{t\in[0,\zeta]} (Ty)(t) = (Ty)(\zeta) \\ &= \int_0^1 G(\zeta,s) F\big(s,y(s)\big)ds + A(f)(b+a\zeta) + B(f)\big(d+c(1-\zeta)\big) \\ &< \frac{h}{\Lambda}\left(\int_0^1 G(\zeta,s)\, ds + A(b+a\zeta) + B\big(d+c(1-\zeta)\big)\right) = h.\end{aligned}$$

So, we get $\varphi(Ty) < h$. Thus, (iii) of Lemma 8 is satisfied.

Since all conditions of Lemma 8 are satisfied, $\mathbb{T}$ has at least three fixed points, y_1, y_2 and y_3 belonging to $\overline{\mathcal{P}(\psi, v)}$ of BVP (2.3) such that

$$0 < \varphi(y_1) < h < \varphi(y_2), \qquad \theta(y_2) < p < \theta(yu_3), \qquad \psi(y_3) < v.$$

Then the nth-order BVP (1.1) has at least three positive solutions

$$u_i(t) = \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y_i(r)dr \quad (i = 1, 2, 3).$$

The proof is complete. $\square$

4 Examples

Example 1. Consider the following problem

$$\begin{cases} u'''(t) + f\big(t, u(t)\big) = 0, \quad t \in [0,1], \\ 2u'(0) - u''(0) = \displaystyle\int_0^1 u'(s)ds, \\ \dfrac{1}{2}u'(1) + \dfrac{9}{4}u''(1) = \displaystyle\int_0^1 u'(s)ds, \\ u(0) = 0, \end{cases} \tag{4.1}$$

where

$$f(t,u) = \begin{cases} \frac{1}{10}(t+37) + \frac{3}{500}u, & (t,u) \in [0,1] \times [0,50], \\ \frac{1}{10}(t + 3606u) - 18026, & (t,u) \in [0,1] \times [50, \infty). \end{cases}$$

By simple calculation, we get $\rho = 6$, $\triangle = -9$, $A = B = \frac{4}{9}$ and

$$G(t,s) = \frac{1}{6} \begin{cases} (1+2s)\left(\frac{11}{4} - \frac{t}{2}\right), & s \le t, \\ (1+2t)\left(\frac{11}{4} - \frac{s}{2}\right), & t \le s. \end{cases}$$

Taking $j = 1$, $k = 10$, $l = 1000$, $\eta = \frac{1}{5}$, $\xi = \frac{1}{2}$, we have $0 < \eta < \xi < 1$, $j < k < l$. Through some simple calculation, we get

$$L = \frac{133}{480}, \quad N = \frac{709}{300},$$
$$0 < j = 1 < \frac{L}{N}k = \frac{3325}{2836} < \frac{L\eta^2(\xi-\eta)^{n-2}}{N(n-2)!}l = \frac{1995}{1418}.$$

It is clear that $(C1)$–$(C6)$ are satisfied. Next, we show that $(C7)$–$(C9)$ are also satisfied.

For $(t,u) \in \left[\frac{1}{2}, 1\right] \times [60, 5000]$, we have $f(t,u) \ge 3610.05 > \frac{l}{L} = \frac{480000}{133}$. So $(C7)$ is satisfied. For $(t,u) \in [0,1] \times [0,50]$, we have $f(t,u) \le 4.1 < \frac{k}{N} = \frac{3000}{709}$. Hence $(C8)$ is satisfied. For $(t,u) \in \left[\frac{1}{2}, 1\right] \times [0,5]$, we get $f(t,u) \ge 3.75 > \frac{j}{L} = \frac{480}{133}$. So $(C9)$ is satisfied. Then all conditions of Theorem 1 hold. Hence, BVP (4.1) has at least two positive solutions.

Example 2. Consider the following problem

$$\begin{cases} u^{(5)}(t) + f\big(t, u(t)\big) = 0, \quad t \in [0,1], \\ 3u^{(3)}(0) - u^{(4)}(0) = \displaystyle\int_0^1 su^{(3)}(s)ds, \\ 2u^{(3)}(1) + u^{(4)}(1) = \displaystyle\int_0^1 9s^2u^{(3)}(s)ds, \\ u(0) = u'(0) = u''(0) = 0, \end{cases} \tag{4.2}$$

where

$$f(t,u) = \begin{cases} \frac{t}{100} + 0.2, & (t,u) \in [0,1] \times [0,4], \\ \frac{t}{100} + 1174999.8u - 4699999, & (t,u) \in [0,1] \times [4,5], \\ \frac{t}{100} + 1175000, & (t,u) \in [0,1] \times [5,\infty). \end{cases}$$

By simple calculation, we get $\rho = 11$, $\triangle = -\frac{143}{24}$, $A = \frac{276969}{173030}$, $B = \frac{86757}{346060}$ and

$$G(t,s) = \frac{1}{11} \begin{cases} (1+3s)(3-2t), & s \le t, \\ (1+3t)(3-2s), & t \le s. \end{cases}$$

Choosing $\mu = \frac{1}{4}$, $\zeta = \frac{1}{3}$, we have $0 < \mu < \zeta < 1$ and $\Omega = \frac{35}{198}$, $\Lambda = \frac{140275421}{34259940}$. Taking $h = 1$, $p = 207360$, $v = 5 \times 10^6$, we get

$$\frac{h(n-2)!}{\mu^2 (\zeta - \mu)^{n-2}} = 165888 < p = 207360 < \frac{\Omega}{\Lambda} v = 215862.83.$$

It is clear that $(C1)$–$(C6)$ are satisfied. Next, we show that $(C10)$–$(C12)$ are also satisfied.

Firstly, for $(t,u) \in [0,1] \times [0, 2 \times 10^7]$, we have $f(t,u) \le 1175000.01 < \frac{v}{\Lambda} = 1221166.893$. So $(C10)$ is satisfied.

For $(t,u) \in [\frac{1}{3}, 1] \times [5, 829440]$, we have $f(t,u) \ge 1175000 > \frac{p}{\Omega} = 1173065.143$. Hence $(C11)$ is satisfied.

Lastly, for $(t,u) \in [0,1] \times [0,4]$, we get $f(t,u) \le 0.21 < \frac{h}{\Lambda} = \frac{34259940}{140275421}$. So $(C12)$ is satisfied.

Then all conditions of Theorem 2 hold. Hence, BVP (4.2) has at least three positive solutions.

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