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# GENERALIZED LINEAR METHODS AND CONVERGENCE ACCELERATION

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**ABSTRACT**

Several $\lambda$-boundedness propositions for generalized linear methods $\mathcal{A} = (A_{nk})$, while $A_{nk}$ are specially fixed linear bounded operators from Banach space $X$ into $X$, are presented. These results are proved using necessary and sufficient conditions for inclusion $\mathcal{A}m_X^\lambda \subset m_X^\mu$.

**Key words:** convergence acceleration, summability methods

## 1. INTRODUCTION

Let $X$, $Y$ be Banach spaces and $\mathcal{L}(X,Y)$ be a space of linear bounded operators from $X$ into $Y$. A sequence $x = (\xi_k)$ $(\xi_k \in X)$ is called $\lambda$-bounded ($\lambda$-convergent) if $\beta_k = O(1)$ $\left(\exists \lim\limits_{k\to\infty} \beta_k\right)$, whereas $\lambda = (\lambda_k)$ with $0 < \lambda_k \nearrow$ and $\beta_k = \lambda_k\,(\xi_k - \xi)$ with $\xi = \lim\limits_{k\to\infty} \xi_k$. Let $m_X^\lambda$ $\left(c_X^\lambda\right)$ be a set of all $\lambda$-bounded ($\lambda$-convergent) sequences. If $\lambda_k = O(1)$, then

$$m_X^\lambda = c_X^\lambda = c_X,$$

while $c_X$ is a set of all convergent sequences $(\xi_k)$ $(\xi_k \in X)$. A sequence $x = (\xi_k)$ is called (see [4] or [13]) summable by a generalized method $\mathcal{A} = (A_{nk})$, $A_{nk} \in \mathcal{L}(X,Y)$ if $y = (\eta_n)$ with

$$\eta_n = \sum_{k=0}^{\infty} A_{nk}\xi_k \tag{1.1}$$

is convergent.

Unless indicated otherwise a sum $\sum_k$ will always be understood as $\sum_{k=0}^{\infty}$ and a limit $\lim_n$ as $\lim_{n\to\infty}$. The transformation $\mathcal{A}$ is called accelerating $\lambda$–boundedness ($\lambda$-convergence) if $Am_X^\lambda \subset m_Y^\mu$ $\left(Ac_X^\lambda \subset c_Y^\mu\right)$ with $0 < \mu_k \nearrow$ and

$$\lim_k \mu_k/\lambda_k = \infty. \tag{1.2}$$

In applied mathematics we often study cases with weaker condition than (1.2). In the sequel $\lambda_k \nearrow \infty$ and $\mu_k \nearrow \infty$. A method $\mathcal{A} = (A_{nk})$ is called triangular if $A_{nk} = \theta$ $(k > n)$, whereas $\theta$ is the zero-operator. A method $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, X)$ is called regular if $\mathcal{A}c_X \subset c_X$ and $\lim_n \eta_n = \lim_k \xi_k$. Let $I \in \mathcal{L}(X, X)$ be the identity operator. In the case of number sequences and matrices Kangro [5] proved that a regular triangular method can not accelerate the convergence. Kornfeld [7] generalized Kangro's result for any regular number-matrix method.

To prove our propositions we use two Lemmas (see [11] and [12]).

**Lemma 1.1.** *Let* $A_{nk} \in \mathcal{L}(X, Y)$, $\mathcal{A} = (A_{nk})$ *and* $e_X(\varsigma) := (\varsigma, \varsigma, \varsigma, \ldots)$ *with* $\varsigma \in X$. *If*

$$\exists \lim_n A_{nk} = A_k \quad (k \in \mathbf{N}_0) \tag{1.3}$$

*in norm, then the conditions*

$$\mathcal{A}e_X(\zeta) \in m_Y^\mu \quad (\zeta \in X), \tag{1.4}$$

$$\sum_k \lambda_k^{-1} \|A_k\| < \infty, \tag{1.5}$$

$$\mu_n \sum_k \lambda_k^{-1} \|A_{nk} - A_k\| = O(1) \tag{1.6}$$

*are necessary and sufficient for the inclusion*

$$\mathcal{A}m_X^\lambda \subset m_Y^\mu. \tag{1.7}$$

**Lemma 1.2.** *Let* $A_{nk} \in \mathcal{L}(X, Y)$, $\mathcal{A} = (A_{nk})$ *and*

$$e_k(\varsigma) := (0, \ldots, 0, \varsigma, 0, \ldots), \quad e_\lambda(\varsigma) := \left(\lambda_1^{-1}\varsigma, \lambda_2^{-1}\varsigma, \lambda_3^{-1}\varsigma, \ldots\right)$$

*with* $\varsigma \in X$. *The conditions*

$$\mathcal{A}e_k(\varsigma) \in c_Y^\mu, \;\; \mathcal{A}e(\varsigma) \in c_Y^\mu, \;\; \mathcal{A}e_\lambda(\varsigma) \in c_Y^\mu,$$

$$\sup_{\|\varsigma_k\| \le 1} \sum_{k=0}^{p} \lambda_k^{-1} \|A_{nk}\varsigma_k\| = O(1) \quad (\varsigma_k \in X, \; n, p \in \mathbf{N}_0),$$

$$\sup_{\|\varsigma_k\|\le 1} \sum_{k=0}^{p} \mu_n \lambda_k^{-1} \|(A_{nk} - A_k)\, \varsigma_k\| = O(1) \quad (\varsigma_k \in X,\ n, p \in \mathbf{N}_0)\,,$$

*are necessary and sufficient for the inclusion* $\mathcal{A}c_X^\lambda \subset c_Y^\mu$.

In this article using Lemma 1.1 several partial cases of $\mathcal{A}m_X^\lambda \subset m_Y^\mu$ are studied. Using Lemma 1.2 analogical results for $\mathcal{A}c_X^\lambda \subset c_Y^\mu$ can be derived.

## 2. CONVERGENCE ACCELERATION USING GENERALIZED RIESZ METHOD

Let us denote by $(\Re, P_n)$, or shortly by $\Re$, the generalized Riesz method with $R_{nk} \in \mathcal{L}(X, X)$, defined in [8] by

$$R_{nk} = \begin{cases} R_n P_k & (k = 0, 1, \dots, n)\,, \\ \theta & (k > n)\,, \end{cases} \tag{2.1}$$

where $P_k$, $R_n \in \mathcal{L}(X, X)$, while $R_n$ is determined by

$$R_n \sum_{k=0}^{n} P_k\, \zeta = \zeta \quad (\zeta \in X,\ n \in \mathbf{N}_0)\,. \tag{2.2}$$

**Lemma 2.1.** [8]. *If*

$$\lim_n \|R_n\| = 0 \tag{2.3}$$

*and*

$$\|R_n\| \sum_{k=0}^{n} \|P_k\| = O(1), \tag{2.4}$$

*then the method* $(\Re, P_n)$ *is regular.*

**Proposition 2.1.** *If* $X$ *is a Banach space, then the conditions* (2.3), (2.4) *and*

$$\mu_n \|R_n\| \sum_{k=0}^{n} \lambda_k^{-1} \|P_k\| = O(1) \tag{2.5}$$

*are sufficient for the inclusion*

$$\Re\, m_X^\lambda \subset m_X^\mu. \tag{2.6}$$

*Proof.* Let us use Lemma 1.1 by fixing $\mathcal{A} = \Re$. By Lemma 2.1 the conditions (2.3) and (2.4) are sufficient for the regularity of the method $\Re$. As $\mathcal{A} = \Re$ is regular, then (see [8] or [9]) $A_k = \theta$ $(k \in \mathbf{N}_0)$. Using (1.1), (2.1) and (2.2) we get

$$\eta_n = \sum_{k=0}^{n} R_n P_k\, \zeta = \zeta \quad (\zeta \in X,\ n \in \mathbf{N}_0)\,.$$

So we have

$$\eta = \lim_n \sum_{k=0}^{n} R_n P_k \, \zeta = \zeta,$$

$$\mu_n \left(\eta_n - \eta\right) = \mu_n \left(\zeta - \zeta\right) = 0 \, .$$

That means

$$\mathcal{A}\, e_X \;(\zeta) \in m_X^{\mu}$$

and the condition (1.4) is satisfied. As $A_k = \theta$ $(k \in \mathbf{N}_0)$ , then the condition (1.5) is satisfied. The condition (1.3) follows from

$$\|A_{nk} - A_k\| = \|R_n P_k - \theta\| \le \|R_n\| \, \|P_k\| \overset{n\to\infty}{\to} 0.$$

The condition (1.6) follows from the condition (2.5). That means the conditions of Lemma 1.1 are satisfied and from (1.7) we get the assertion (2.6). This completes the proof. ■

Using [12] we get the following result.

*Remark 2.1.* If the conditions of the Proposition 2.1 are satisfied, then $\mu_n/\lambda_n = O(1)$, that means a generalized Riesz method $\Re$, satisfying the conditions of Proposition 2.1, can not accelerate the convergence.

*Remark 2.2.* If we want to use a method $\Re$ for acceleration of the convergence we have to use nonregular methods. A regular method $\Re$ is used to accelerate the convergence in some subsets of $m_X^{\lambda}$ (see [10]).

## 3. CONVERGENCE ACCELERATION USING GENERALIZED EULER-KNOPP METHOD

Let us denote by $(\mathcal{E}, \Lambda)$ , or shortly by $\mathcal{E}$, the generalized Euler-Knopp method with $E_{nk} \in \mathcal{L}(X,X)$, defined in [8] by

$$E_{nk} = \begin{cases} \binom{n}{k} \Lambda^k \left(I - \Lambda\right)^{n-k} & (k = 0, 1, \ldots, n)\,, \\ \theta \quad (k > n)\,, \end{cases} \tag{3.1}$$

where $\Lambda \in \mathcal{L}(X,X)$, $\Lambda \ne \theta$ and $\Lambda^0 = I$.

**Lemma 3.1.** [8]. *Method $(\mathcal{E}, \Lambda)$ is regular if and only if*

$$\|\Lambda\| + \|I - \Lambda\| \le 1 \tag{3.2}$$

*and*

$$\|I - \Lambda\| < 1. \tag{3.3}$$

**Proposition 3.1.** *If $X$ is a Banach space, then the conditions* (3.2), (3.3) *and*

$$\mu_n \sum_{k=0}^{n} \lambda_k^{-1} \binom{n}{k} \|\Lambda\|^k = O(1) \tag{3.4}$$

*are sufficient for the inclusion*

$$\mathcal{E}\, m_X^{\lambda} \subset m_X^{\mu}. \tag{3.5}$$

*Proof.* Let us verify the conditions of Lemma 1.1 by fixing $\mathcal{A} = \mathcal{E}$. By Lemma 3.1 the conditions (3.2) and (3.3) are sufficient for the regularity of the method $\mathcal{E}$. As $\mathcal{E}$ is regular, then (see [8]) $A_k = \theta \ (k \in \mathbf{N}_0)$. The condition (1.5) follows from $A_k = \theta \ (k \in \mathbf{N}_0)$. Using (1.1) and (3.1) we get for $\zeta \in X$ that

$$\eta_n = \sum_{k=0}^{n} \binom{n}{k} \Lambda^k (I - \Lambda)^{n-k} \zeta = (\Lambda + (I - \Lambda))^n \zeta$$
$$= I\varsigma = \zeta \quad (n \in \mathbf{N}_0) .$$

So we have

$$\eta = \lim_n \eta_n = \lim_n \zeta = \zeta ,$$
$$\mu_n (\eta_n - \eta) = \mu_n (\zeta - \zeta) = 0 ,$$

and

$$\mathcal{A}\, e_X \ (\zeta) \in m_X^{\mu} \quad (\zeta \in X ) .$$

That means the condition (1.4) is satisfied. As $A_k = \theta \ (k \in \mathbf{N}_0)$, by condition (3.3) we get

$$\|A_{nk} - A_k\| = \|A_{nk}\| = \left\| \binom{n}{k} \Lambda^k (I - \Lambda)^{n-k} \right\|$$
$$\leq \binom{n}{k} \|\Lambda\|^k \|(I - \Lambda)\|^{n-k} \overset{n \to \infty}{\to} 0.$$

So the condition (1.3) is satisfied. The condition (1.6) follows from condition (3.4). So the conditions of Lemma 1.1 are satisfied and from (1.7) we get (3.5). This completes the proof. ■

Using [12] we get the following result.

*Remark 3.1.* If the conditions of the Proposition 3.1 are satisfied, then $\dfrac{\mu_n}{\lambda_n} = O(1)$, that means a generalized Euler-Knopp method $\mathcal{E}$, satisfying the conditions of Proposition 3.1 can not accelerate the convergence.

*Remark 3.2.* If we want to use a method $\mathcal{E}$ for acceleration of the convergence we have to use nonregular methods. A regular method $\mathcal{E}$ is used to accelerate the convergence in some subsets of $m_X^\lambda$.

## 4. CONVERGENCE ACCELERATION USING PROJECTIONS

Let $\{T_k\}$ $(k \in \mathbf{N}_0)$ be a sequence of operators, while $T_k \in \mathcal{L}(X, X)$. For example we can use mutually orthogonal continuous projections $T_k$ on Banach space $X$ as it was done in [1]. Let us define generalized method $\mathcal{K} = (K_{nk})$ by

$$K_{nk} := c_{nk} T_k \quad (c_{nk} \in \mathbf{R} \ \vee \ c_{nk} \in \mathbf{C}) \,. \tag{4.1}$$

**Proposition 4.1.** *The conditions*

$$\lim_n c_{nk} = 0, \tag{4.2}$$

$$\mu_n \left( \sum_k c_{nk} T_k \zeta - \lim_n \sum_k c_{nk} T_k \zeta \right) = O(1) \quad (\zeta \in X) \tag{4.3}$$

*and*

$$\mu_n \sum_k \lambda_k^{-1} |c_{nk}| \, \|T_k\| = O(1) \tag{4.4}$$

*are necessary and sufficient for the inclusion*

$$\mathcal{K}\, m_X^\lambda \subset m_X^\mu. \tag{4.5}$$

*Proof.* Let us use Lemma 1.1 taking $\mathcal{A} = \mathcal{K}$. As $T_k$ are bounded operators we get using (4.1) and (4.2) that $A_k = \theta$ and likewise that the conditions (1.3) and (1.5) are satisfied. The condition (1.4) follows from (4.3) and (1.6) follows from (4.4). So all the conditions of the Lemma 1.1 are satisfied and (4.5) follows from (1.7). ■

*Remark 4.1.* Using Proposition 4.1 and the results proved in [1] several convergence acceleration theorems for the method of Riesz (see [2]), for the method of Jackson-de La Vallée Poussin (see [2]), for the method of Bohman-Korovkin (see [3]), for the method of Zhuk (see [14]) and for the method of Favard (see [6]) can be derived.

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## Apibendrinti tiesiniai metodai ir konvergavimo greitinimas

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Straipsnyje pateiktos kelios $\lambda$ – aprėžtumo teoremos apibendrintiems tiesiniams metodams $\mathcal{A} = (A_{nk})$, kur $A_{nk}$ yra tam tikri fiksuoti tiesiniai aprėžti operatoriai, apibrėžti Banacho erdvėje $X$ $X$. Teoremos įrodytos naudojantis būtinomis ir pakankamomis sąlygomis $\mathcal{A}m_X^\lambda \subset m_X^\mu$.