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RELAXATION OF A WEAKLY DISCONTINUOUS FUNCTIONAL DEPENDING ON ONE CONTROL FUNCTION

U. RAITUMS

Institute of Mathematics and Computer Science, University of Latvia

Rainis b. 29, LV-1459, Riga, Latvia

E-mail: uldis.raitums@mii.lu.lv

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Abstract. The paper considers an optimal control problem of the type

$$\begin{cases} \mathcal{J} = \int_\Omega [\langle B(x)\nabla u,\, \nabla u\rangle + \langle g, \nabla u\rangle] dx \to \min, \\ \operatorname{div}[A(x)\nabla u - h(x)] = 0 \quad \text{in } \Omega,\ u \in H_0^1(\Omega; \mathbb{R}^m),\ h \in \mathcal{M}, \end{cases}$$

where the set $\mathcal{M}$ of admissible controls consists of all measurable vector-functions h, which can take only two values h_1 or h_2. It is shown that the relaxation of this problem can be explicitly computed by rank-one laminates.

Key words: optimal control, elliptic system, weakly discontinuous functional, relaxation.

1 Introduction

We consider the optimal control problem

$$\begin{cases} \mathcal{J} = \int_\Omega [\langle B(x)\nabla u,\, \nabla u\rangle + \langle g(x),\, \nabla u\rangle]\, dx \to \min, \\ \operatorname{div}[A(x)\nabla u - f(x) - \sigma(x)h(x)] = 0 \quad \text{in } \Omega, \\ u = (u_1, \dots, u_m) \in H_0^1(\Omega; \mathbb{R}^m), \\ \sigma \in \mathcal{S} = \{\sigma \in L_{2loc}(\mathbb{R}^n) | \sigma(x) = 0 \text{ or } 1\}, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^n$ is a given bounded Lipschitz domain; A and B are given piecewise constant symmetric $nm \times nm$ matrices, A positive definite; f and h are given functions from $L_{2loc}(\mathbb{R}^n; \mathbb{R}^{nm})$; the function σ plays the role of the control.

The main features of (1.1) are that the state equation is given by an elliptic system and that the matrix $B(\cdot)$ is arbitrary, what gives that the resulting mapping $\sigma \mapsto \mathcal{J}(u(\sigma))$, $u(\sigma)$ being the solution of the state equation corresponding to σ, is not weakly continuous or weakly lower semicontinuous.

Problems with weakly discontinuous functionals were studied mostly in the context of optimal design problems and, more or less, explicit formulae for relaxed problems were obtained for cases with only one control function-characteristic function of the domain occupied by one of two materials, see, e.g., [2, 3, 4]. For these cases, the laminated structures give the relaxation of the problem. It appears that laminated structures play an analogous role for the problem (1.1).

We would like to mention here that one of sources for the problem (1.1) is the question of evaluation of the second order terms in optimal design problems. An analogue of the problem of minimal stiffness (see [1]) can be written as

$$\begin{cases} \inf\{\mathcal{J}(\theta,\, \sigma) | \theta \in [0,\, 1], \sigma \in \mathcal{S}\}, \\ \mathcal{J}(\theta,\, \sigma) = \inf\limits_{u \in H_0^1(\Omega;\, \mathbb{R}^m)} \displaystyle\int_\Omega [\langle (A + \theta\sigma(x)\delta A)(\nabla u + g_1(x)),\, \nabla u + g_1(x)\rangle \\ \qquad\qquad\qquad\qquad\qquad - 2\langle g_2(x),\, \nabla u\rangle]\, dx, \end{cases}$$

with given A, δA, g_1, g_2. For this problem, the second derivative $\mathcal{J}''_{\theta\theta}(0,\sigma)$ has the representation

$$\mathcal{J}''_{\theta\theta}(0,\sigma) = -\int_\Omega \langle A\nabla v,\, \nabla v\rangle\, dx,$$
$$\operatorname{div}[A\nabla v + \sigma(x)\delta A(\nabla u_0(x) + g_1(x))] = 0 \quad \text{in } \Omega,\, v \in H_0^1(\Omega;\, \mathbb{R}^m),$$

where $u_0 \in H_0^1(\Omega;\, \mathbb{R}^m)$ satisfies

$$\operatorname{div}[A(\nabla u_0 + g_1(x)) - g_2(x)] = 0 \quad \text{in } \Omega.$$

Obviously, the problem of minimizing $\mathcal{J}''_{\theta\theta}(0,\sigma)$ over $\sigma \in \mathcal{S}$ is the same as the problem (1.1) with $B = -A$ being negative.

2 Evaluation of Formal Relaxation

It could be shown, a sketch of proofs is given in Appendix, that the relaxation of (1.1) is the joint passage from $\mathcal{S}$ to its closed convex hull $\overline{\text{co}}\mathcal{S}$ and from $\mathcal{J}$ to $\tilde{\mathcal{J}}$,

$$\tilde{\mathcal{J}}(u,\, \hat{\sigma}) = \mathcal{J}(u) + \int_\Omega F(x, \hat{\sigma}(x))\, dx$$

with $\hat{\sigma} \in \overline{\text{co}}\mathcal{S}$, u satisfying the state equation in (1.1) with $\sigma = \hat{\sigma} \in \overline{\text{co}}\mathcal{S}$ and the function F defined as

$$\begin{aligned} F(x_0, \hat{\sigma}) &= \inf_{\sigma \in \mathcal{S},\, \int_K \sigma(x)\,dx = \hat{\sigma}} J(x_0, \hat{\sigma}, \sigma), \\ J(x_0, \hat{\sigma}, \sigma) &= \int_K \langle B(x_0)\nabla v, \nabla v\rangle\,dx \quad \text{with} \\ &\qquad \operatorname{div}[A(x_0)\nabla v - (\sigma(x) - \hat{\sigma})h(x_0)] = 0 \quad \text{in } K, \\ &\qquad v \in H^1_{loc}(\mathbb{R}^n; \mathbb{R}^m), v \text{ is } K\text{-periodic}, \end{aligned} \tag{2.1}$$

where K is the unit cube $(0, 1)^n$ and $\hat{\sigma} \in [0, 1]$.

We want to evaluate F from below. Let

$$\begin{aligned} \mathbb{L} = \{l \in \mathbb{Z}^n | l = (l_1, \ldots, l_n),& \\ l_1 > 0 \text{ or } l_1 = \cdots = l_{i-1} = 0 \;\&\; l_i > 0 &\text{ for some } i = 2, \ldots, n\}. \end{aligned}$$

Obviously, the system $\{\sqrt{2}\sin 2\pi\langle x, l\rangle, \sqrt{2}\cos 2\pi\langle x, l\rangle | l \in \mathbb{L}\}$ is complete and orthonormal in $L_2(K)/\mathbb{R}$ and the system $\{\sin 2\pi\langle x, l\rangle l, \cos 2\pi\langle x, l\rangle l | l \in \mathbb{L}\}$ is complete and orthogonal in $H^{\#}$,

$$H^{\#} = \{w \in L_2(K; \mathbb{R}^{nm}) | w = \nabla v,\ v \in H^1_{loc}(\mathbb{R}^n; \mathbb{R}^m),\ v \text{ is } K\text{-periodic}\}.$$

Since $A(x_0)$ is positive definite and $A(x_0)$, $B(x_0)$, $h(x_0)$ do not depend on $x \in K$, then expanding in (2.1) the unknown function v in the Fourier series we easily obtain

$$\begin{aligned} J(x_0, \hat{\sigma}, \sigma) \;&= \sum_{l \in \mathbb{L}} \Big\{ \langle A_l^{-1*} B_l A_l^{-1} q_l, q_l\rangle \\ \times \Big[\Big(\int_K (\sigma(x) - \hat{\sigma})&\sqrt{2}\sin 2\pi\langle x, l\rangle\,dx\Big)^2 + \Big(\int_K (\sigma(x) - \hat{\sigma})\sqrt{2}\cos 2\pi\langle x, l\rangle dx\Big)^2\Big]\Big\}, \end{aligned} \tag{2.2}$$

where

$$A_l = \begin{pmatrix} \langle A_{11}l, l\rangle \cdots \langle A_{1m}l, l\rangle \\ \vdots \quad \ddots \quad \vdots \\ \langle A_{m1}l, l\rangle \cdots \langle A_{mm}l, l\rangle \end{pmatrix} \text{ for block-matrix } A(x_0) = \begin{pmatrix} A_{11} \cdots A_{1m} \\ \vdots \ \ddots \ \vdots \\ A_{m1} \cdots A_{mm} \end{pmatrix},$$

$$B_l = \begin{pmatrix} \langle B_{11}l, l\rangle \cdots \langle B_{1m}l, l\rangle \\ \vdots \quad \ddots \quad \vdots \\ \langle B_{m1}l, l\rangle \cdots \langle B_{mm}l, l\rangle \end{pmatrix} \text{ for block-matrix } B(x_0) = \begin{pmatrix} B_{11} \cdots B_{1m} \\ \vdots \ \ddots \ \vdots \\ B_{m1} \cdots B_{mm} \end{pmatrix},$$

$$q_l = \begin{pmatrix} \langle h^1, l\rangle \\ \vdots \\ \langle h^m, l\rangle \end{pmatrix} \text{ for } h(x_0) = \begin{pmatrix} h^1 \\ \vdots \\ h^m \end{pmatrix} = \left(h^1_1, \ldots, h^1_n, \ldots, h^m_1, \ldots, h^m_n\right)^T. \tag{2.3}$$

The sequence $\{\langle A_l^{-1*}B_lA_l^{-1}q_l,\, q_l\rangle\}$ is bounded, hence, there exists $l_0 \in \mathbb{R}^n$, $|l_0| = 1$, such that

$$\langle A_l^{-1*}B_lA_l^{-1}q_l,\, q_l\rangle \geq \langle A_{l_0}^{-1*}B_{l_0}A_{l_0}^{-1}q_{l_0},\, q_{l_0}\rangle \quad \forall l \in \mathbb{L},$$

where A_{l_0}, B_{l_0}, q_{l_0} are computed by the same formulae (2.3) with $l = l_0$.

From here, (2.2) and completeness of the chosen basis system in $L_2(K)/\mathbb{R}$ it follows immediately

$$F(x_0,\, \hat{\sigma}) \geq \inf\{\langle A_l^{-1*}B_lA_l^{-1}q_l,\, q_l\rangle \int_K (\sigma(x) - \hat{\sigma})^2\, dx| \\ \sigma \in S,\, \int_K \sigma(x)\, dx = \hat{\sigma},\, l \in \mathbb{R}^n,\, |l| = 1\}, \quad (2.4)$$

where A_l, B_l, q_l are computed by (2.3).

3 Laminated Structure

We want to show that for a fixed $\hat{\sigma} \in [0,\, 1]$ the value in the right hand side of (2.4) can be attained by values of $J(x_0,\, \hat{\sigma},\, \sigma)$ with functions $\sigma \in S$ depending on only one direction, i.e. functions $\sigma \in S$ of the type $\sigma = \sigma(\langle x, l_*\rangle)$ with $l_* \in \mathbb{Q}^n$. For fixed x_0 the mapping

$$l \mapsto \langle A_l^{-1*}B_lA_l^{-1}q_l,\, q_l\rangle$$

is continuous on $\mathbb{R}^n/\{0\}$ and for every chosen $\hat{\sigma} \in [0,\, 1]$, $\sigma \in S$, $l_* \in \mathbb{Q}^n$, $l_* \neq 0$ there exists $l_0 \in \mathbb{Z}^n$ and $\sigma_0 \in S$ such that

$$\begin{aligned} &l_0 \neq 0,\, l_0 \parallel l_*,\, \sigma_0 = \sigma_0(\langle x,\, l_0\rangle),\, \int_K [\sigma_0(\langle x,\, l_0\rangle) - \sigma(x)]\, dx = 0, \\ &\int_K (\sigma_0(\langle x,\, l_0\rangle) - \hat{\sigma})^2\, dx = \int_K (\sigma(x) - \hat{\sigma})^2\, dx, \\ &\langle A_{l_0}^{-1*}B_{l_0}A_{l_0}^{-1}q_{l_0},\, q_{l_0}\rangle = \langle A_{l_*}^{-1*}B_{l_*}A_{l_*}^{-1}q_{l_*},\, q_{l_*}\rangle. \end{aligned}$$

The function σ_0 is K-periodic, therefore, the boundary value problem

$$\begin{cases} \operatorname{div}[A(x_0)\nabla v - (\sigma_0(\langle x,\, l_0\rangle) - \hat{\sigma})h(x_0)] = 0 \quad \text{in } K, \\ v \in H^1_{loc}(\mathbb{R}^n;\, \mathbb{R}^m),\ v \text{ is } K\text{-periodic}, \end{cases}$$

has a solution $v_0 = v_0(\langle x,\, l_0\rangle)$, which can be computed exactly, and straightforward computations give

$$\int_K \langle B(x_0)\nabla v_0, \nabla v_0\rangle\, dx = \langle A_{l_0}^{-1*}B_{l_0}A_{l_0}^{-1}q_{l_0},\, q_{l_0}\rangle \int_K (\sigma_0(\langle x,\, l_0\rangle) - \hat{\sigma})^2\, dx.$$

From here, (2.1) and (2.4) it follows immediately

$$F(x_0,\hat\sigma) = \inf\{\langle A_l^{-1*}B_lA_l^{-1}q_l,\, q_l\rangle \int_K (\sigma(\langle x,\, l\rangle) - \hat\sigma)^2\,dx| \\ \sigma\in S, \sigma=\sigma(\langle x,l\rangle), \int_K \sigma(\langle x,\, l\rangle)\,dx = \hat\sigma,\, l\in\mathbb{R}^n,\, |l|=1\} \\ = \min_{l\in\mathbb{R}^n,\,|l|=1} \langle A_l^{-1*}B_lA_l^{-1}q_l,\, q_l\rangle \hat\sigma(1-\hat\sigma). \quad (3.1)$$

Here the value of the inner infimum over σ follows from the fact that σ takes only two values. The (3.1) gives the exact formula (computable by standard minimization procedure in $\mathbb{R}^n$) for relaxation of (1.1) and (3.1) was obtained by means of rank-one laminates.

4 Appendix

From what was proved in [5] in the context of G-convergence for the scalar case with nonlinear first order terms of elliptic operators (all reasoning remains valid for the vectorial case too) we have the following. If

$$\begin{aligned} &\operatorname{div}[A(x)\nabla u_k - h_k(x)] = 0 \quad \text{in } \Omega, && k=0,\,1,\,2,\,3,\,\ldots, \\ &u_k\in H_0^1(\Omega;\,\mathbb{R}^m),\ h_k\in L_{n+1}(\Omega;\,\mathbb{R}^{nm}), && k=0,\,1,\,2,\,3,\,\ldots, \\ &h_k \rightharpoonup h_0 \quad \text{weakly as } k\to\infty, \end{aligned} \quad (4.1)$$

then for appropriate functions $f=f(x,\,z)$, $x\in\Omega$, $z\in\mathbb{R}^{nm}$ (the function

$$f_0(x,\,z) = \langle B(x)z,\,z\rangle + \langle g(x),\,z\rangle$$

is eligible), there exists a subsequence $\{h_s\}\subset\{h_k\}$ such that

$$\int_\Omega f(x,\,\nabla u_s)\,dx \to \int_\Omega \tilde f(x,\,\nabla u_0)\,dx \quad \text{as } k\to\infty, \quad (4.2)$$

where for a.e. $x_0\in\Omega$

$$\begin{aligned} &\tilde f(x_0,\,z) = \lim_{\tau\to0}\lim_{s\to\infty}\int_K f(x_0,\,z+\nabla v_s(y))\,dy, \\ &\operatorname{div}[A(x_0)\nabla v_s(y) - (h_s(x_0+\tau y) - h_s(x_0))] = 0 \quad \text{in } K, \\ &v_s\in H_{loc}^1(\mathbb{R}^n;\,\mathbb{R}^m),\ v_s \text{ is } K\text{-periodic},\ s=1,\,2,\,\ldots. \end{aligned} \quad (4.3)$$

For our case of (1.1) with $\sigma_k \rightharpoonup \hat\sigma$ weakly in $L_2(\Omega)$

$$\begin{aligned} &\tilde f_0(x_0,\,z) = \langle B(x_0)z, z\rangle + \langle g(x_0),\,z\rangle + f_1(x_0,\,\hat\sigma(x_0)), \\ &f_1(x_0,\,z) = \lim_{\tau\to0}\lim_{s\to\infty}\int_K \langle B(x_0)\nabla v_{\tau s}(y),\,\nabla v_{\tau s}(y)\rangle\,dy, \\ &\operatorname{div}[A(x_0)\nabla v_{\tau s}(y) - (\sigma_s(x_0+\tau y) - \hat\sigma(x_0))h(x_0)] = 0 \quad \text{in } K, \\ &v_{\tau s}\in H_{loc}^1(\mathbb{R}^n;\,\mathbb{R}^m),\ v_{\tau s} \text{ is } K\text{-periodic}, \end{aligned} \quad (4.4)$$

with some subsequence $\{\sigma_s\}\subset\{\sigma_k\}$.

For a fixed $\tau > 0$ the sequence of functions $\{\sigma_s(x_0 + \tau \cdot)\}$ belongs to S and

$$\int_K \sigma_s(x_0 + \tau y)\, dy \to \hat{\sigma}(x_0) \quad \text{as } s \to \infty.$$

Obviously, there exists a sequence $\{\sigma_s^{'}\} \subset S$ such that

$$\int_K \sigma_s^{'}(y) dy = \hat{\sigma}(x_0) \quad \text{and} \quad \|\sigma_s^{'} - \sigma_s\|_{L_2(K)} \to 0 \quad \text{as } s \to \infty.$$

That gives that for a fixed sequence $\{\sigma_k\}$ $f_1(x_0,\, \hat{\sigma}(x_0)) \geq F(x_0,\, \hat{\sigma}(x_0))$ (we point out here that even for a fixed $\{\sigma_k\}$ and $\hat{\sigma}$ there could be various subsequences $\{\sigma_s\} \subset \{\sigma_k\}$, which give various values $f_1(x_0,\, \hat{\sigma}(x_0))$).

From (4.1)–(4.4) it follows that the infimum in (1.1) is equal to the value

$$\int_\Omega [\langle B(x)\nabla u_0,\, \nabla u_0\rangle + \langle g(x),\, \nabla u_0\rangle + f_1(x,\, \hat{\sigma}_0(x))]\, dx \tag{4.5}$$

with some $\hat{\sigma}_0 \in \overline{\text{co}} S$ and u_0 being the solution of the state equation in (1.1) with $\sigma = \hat{\sigma}_0$. Here the control $\hat{\sigma}_0$ is defined by some minimizing sequence $\{\sigma_k\}$ for the original problem (1.1) and the function f_1 is defined by (4.4).

The function F defined by (2.1) is piecewise constant with respect to x_0 and the simple structure of S ensure that the mapping $\hat{\sigma} \mapsto F(x_0,\, \hat{\sigma})$ is continuous on $[0,\, 1]$. Therefore, we can evaluate the infimum in the relaxed problem by using only piecewise constant functions $\hat{\sigma}$. Then (working in subsets where A, B, g, $\hat{\sigma}$ are constant) we have that the value $F(x_0,\, \hat{\sigma}(x_0))$ can be approximated with as good precision ε as we like by some $\sigma \in S$ with the mean value over K equal to $\hat{\sigma}(x_0)$. Expanding this function σ via periodicity to the whole $\mathbb{R}^n$ and applying the transform of co-ordinates $x \to sy$ we will have a sequence $\{\sigma_s\} \subset S$, which converges weakly to $\hat{\sigma}(x_0)$, and simple arguments from the theory of homogenization together with estimates via the duality principle will give that for this sequence $\{\sigma_s\}$ the limit

$$\lim_{s\to\infty} \int_K \langle B(x_0)\nabla v_{\tau s}(y),\, \nabla v_{\tau s}(y)\rangle\, dy$$

in (4.4) does not depend on τ (and on the choice of x_0 in the corresponding subset) and is equal to $F(x_0,\, \hat{\sigma}(x_0))$ (with our precision ε). We point out here that, in general, the sequences $\{v_{\tau s}\}$ do not converge. As an analogue we can mention the sequences of saw-tooth functions.

This way, the infimum in both problems (the original problem (1.1) and the relaxed problem for the functional $\tilde{\mathcal{J}}$) is one and the same and for the control $\hat{\sigma}_0$ from (4.5) there is $f_1(x_0,\, \hat{\sigma}_0(x_0)) = F(x_0,\, \hat{\sigma}_0(x_0))$ a.e. $x_0 \in \Omega$. That gives that the relaxed problem has a solution $\hat{\sigma}_0$.

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