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STRONG μ-FASTER CONVERGENCE AND STRONG μ-ACCELERATION OF CONVERGENCE BY REGULAR MATRICES

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Abstract. The present paper continues the study of acceleration of convergence started in the paper [A. Aasma, Proc. Estonian Acad. Sci. Phys. Math., 2006, 55, 4, 195-209]. The new, non-classical convergence acceleration concept, called strong μ-acceleration of convergence (μ is a positive monotonically increasing sequence), is introduced. It is shown that this concept allows to compare the speeds of convergence for a larger set of sequences than the classical convergence acceleration concept. Regular matrix methods are used to accelerate the convergence of sequences.

Key words: Convergence acceleration, matrix methods, speed of convergence.

1 Introduction

The present paper continues the study of acceleration of convergence of real or complex sequences started in [1]. Therefore all the notions not defined in this paper can be found in [1]. Throughout the paper we assume that indices and summation indices are integers, changing from 0 to ∞, if not specified otherwise.

Classically the convergence acceleration is defined as follows (cf. [5, 6]).

Definition 1. Let $x = (x_k)$ and $y = (y_k)$ be convergent sequences with limits ς and ξ, respectively. If

$$\lim_n \frac{|y_n - \xi|}{|x_n - \varsigma|} = 0, \tag{1.1}$$

then it is said that y converges faster than x.

Definition 2. The sequence transformation $T : x \to y$ is said to accelerate the convergence of the sequence x if y converges faster than x.

Some methods, alternative to the classical concept of estimation and comparison of speeds of convergence of sequences, are used in [3, 4]. In [1] another alternative method is proposed, where the concept, called μ-faster convergence (μ is a positive monotonically increasing sequence) is introduced. It is shown that this concept allows the comparison of speeds of convergence for a larger set of sequences than the classical concept and this comparison is more precise.

Let $\mathbf{A} = (a_{nk})$ be a matrix with real or complex entries. A sequence $x = (x_k)$ is said to be $\mathbf{A}$-*summable* if the sequence $\mathbf{A}\,x = (\mathbf{A}_n\,x)$ is convergent, where

$$\mathbf{A}_n x = \sum_k a_{nk} x_k.$$

We denote the set of all $\mathbf{A}$-summable sequences by $c_{\mathbf{A}}$. Thus, a matrix $\mathbf{A}$ determines the summability method on $c_{\mathbf{A}}$, which we also denote by $\mathbf{A}$.

A method $\mathbf{A}$ is said to be *regular* if for each $x = (x_n) \in c$, where c is the set of all convergent sequences, the equality $\lim_n \mathbf{A}_n x = \lim_n x_n$ holds. The convergence acceleration and μ-acceleration of convergence by regular matrix methods were studied correspondingly in [3, 4, 5, 6, 7] and [1].

In the present paper the concept of strong μ-faster convergence is defined and compared with the usual faster convergence concept, determined by Definitions 1 and 2. It is shown here that the new concept allows the comparison of speeds of convergence for a larger set of sequences than the classical concept and this comparison is more precise. It is also proved that if for a sequence $x = (x_n)$ with the limit ς the sequence of absolute differences $(|x_n - \varsigma|)$ is monotonically decreasing, then the strong μ-faster convergence of a sequence y with respect to x coincides with the usual faster convergence of y with respect to x. Also the concept of strong μ-acceleration of convergence by a regular matrix method is defined and it's properties are studied.

2 Main Results

Let φ be a set of sequences such that

$$\varphi = \{x = (x_k) \mid \; x_k = \text{const}, \text{ if } \; k > k_0\}$$

for some $k_0 \geq 0$. For every sequence $x \in c \setminus \varphi$ we denote

$$\mu_x = \{\mu = (\mu_n) \mid \; 0 < \mu_n \nearrow \infty, \; l_n = \mu_n \, |x_n - \lim{}_n x_n| = \mathcal{O}(1), \; l_n \neq o(1)\, \}.$$

Let us remind some notions from [1]. The sequence μ is called a speed of convergence of x. A sequence $\mu^* = (\mu_n^*) \in \mu_x$ is called the limit speed of convergence of x if for all $\mu = (\mu_n) \in \mu_x$ the relation $\mu_n/\mu_n^* = \mathcal{O}(1)$ holds. The limit speed of convergence $\mu^* = (\mu_n^*)$ of a sequence y is said to be higher than the limit speed of convergence $\lambda^* = (\lambda_n^*)$ of a sequence x if the ratio λ_n^*/μ_n^* is upper-bounded, but not lower-bounded. It is said that a sequence y converges μ-faster than x if the limit speed of convergence of y is higher than the limit speed of convergence of x or $y \in \varphi$ and x does not belong to φ.

Now we introduce the concept of strong μ-faster convergence.

Definition 3. Let $\lambda^* = (\lambda_n^*)$ and $\mu^* = (\mu_n^*)$ be correspondingly the limit speeds of convergence of convergent sequences x and y. We say that y converges strongly μ-faster than x, if $\lambda_n^*/\mu_n^* \longrightarrow 0$ or $y \in \varphi$ and x does not belong to φ.

Remark 1. It is easy to see that if y converges strongly μ-faster than x, then y converges also μ-faster than x, but not vice versa. If $y = (y_n)$ converges strongly μ-faster than $x = (x_n)$, then $\lambda_n^* |y_n - \xi| = o(1)$, where $\lambda^* = (\lambda_n^*)$ is the limit speed of x and ξ is the limit of y. But for the case, if y converges only μ-faster, but not strongly μ-faster than x, there exists a subsequence (y_{k_n}) of y so that $\lambda_n^* |y_{k_n} - \xi| \neq o(1)$.

It was proved in [1] that if a sequence $y = (y_n) \in c$ converges faster than $x = (x_n) \in c \setminus \varphi$, then y converges also μ-faster than x. We prove that the similar property holds for the concept of strong μ-faster convergence.

Theorem 1. *If a sequence $y = (y_n) \in c$ converges faster than $x = (x_n) \in c \setminus \varphi$, then y converges also strongly μ-faster than x.*

Proof For $y \in \varphi$ the assertion of Theorem 1 is clearly true. Thus, suppose that $y \in c \setminus \varphi$ converges faster than $x \in c \setminus \varphi$, i.e. relation (1.1) holds, and show that then y converges also strongly μ-faster than x. By Corollary 2.1 of [1] there exists the limit speed of convergence $\lambda^* = (\lambda_n^*) \in \lambda_x$ of x. Using relation (1.1) we have now

$$\lim_n \frac{\lambda_n^* |y_n - \xi|}{\lambda_n^* |x_n - \varsigma|} = 0.$$

Consequently, by Proposition 2.1 from [1] there exists $\vartheta = (\vartheta_n)$, $0 < \vartheta_n \nearrow \infty$, such that

$$\vartheta_n \frac{\lambda_n^* |y_n - \xi|}{\lambda_n^* |x_n - \varsigma|} = \mathcal{O}(1).$$

Denoting $\vartheta_n \lambda_n^* = \mu_n$, we get from the last relation that $\mu_n |y_n - \xi| = \mathcal{O}(1)$ with $0 < \mu_n \nearrow \infty$ and $\mu_n/\lambda_n^* \longrightarrow \infty$. Consequently for the limit speed of convergence $\mu^* = (\mu_n^*)$ of y we have $\mu_n^*/\lambda_n^* \longrightarrow \infty$. Thus y converges strongly μ-faster than x by Definition 3. ■

The opposite assertion to Theorem 1, however, is not valid.

Example 1. Let $x = (x_n) \in c \setminus \varphi$ be given by the relations

$$x_n = \frac{1}{(n+1)2^n} \text{ if } n = 3k,$$

$$(n+1)^3 8^n x_n = o(1) \text{ if } n = 3k+1,$$

$$2^n (n+1)^2 x_n \neq O(1), \;\; 2^n (n+1) x_n = o(1) \text{ if } n = 3k+2,$$

where $k = 0, 1, \ldots$. It was proved in [1] that applying Aitken's process to the subsequence (x_{3k}) of x we get the sequence $y = (y_n)$, where

$$y_n = \frac{9}{8^n \left(1323n^3 + 6993n^2 + 12024n + 6736\right)}.$$

It is easy to see now that y converges not faster than x and x converges not faster than y, but y converges strongly μ-faster than x. Indeed, we can determine the limit speeds of convergence of x and y respectively by $\lambda^* = (\lambda_n^*)$ and $\mu^* = (\mu_n^*)$, where

$$\lambda_n^* = 2^n(n+1), \quad \mu_n^* = 2^{3n}(n+1)^3.$$

As $\mu_n^*/\lambda_n^* \longrightarrow \infty$, then y converges strongly μ-faster than x by Definition 3.

Suppose now that $x = (x_n) \in c \setminus \varphi$ with the limit ς be a sequence for which the sequence of absolute differences $(|x_n - \varsigma|)$ is monotonically decreasing. We show that in this case the strong μ-faster convergence coincides with the classical faster convergence.

Theorem 2. *Let $x = (x_n) \in c \setminus \varphi$ be a sequence (with the limit ς), for which the sequence of absolute differences $(|x_n - \varsigma|)$ is monotonically decreasing. If a sequence $y = (y_n)$ (with limit ξ) converges strongly μ-faster than x, then y converges also faster than x.*

Proof It is not difficult to see that the limit speed $\lambda^* = (\lambda_n^*)$ of a sequence x can be defined by the equality $\lambda_n^* = 1/|x_n - \varsigma|$. If $\mu^* = (\mu_n^*)$ is the limit speed of y, then we get

$$\frac{\mu_n^* \, |y_n - \xi|}{\lambda_n^* \, |x_n - \varsigma|} = \mu_n^* \, |y_n - \xi| = \mathcal{O}(1).$$

Last relation implies equality (1.1), since $\mu_n^*/\lambda_n^* \longrightarrow \infty$. ■

It is said (see [1]) that a regular method $\mathbf{A}$ *μ-accelerates the convergence* of a sequence $x \in c$ if the sequence $\mathbf{A}x$ converges μ-faster than x.

Definition 4. We say that a matrix method $\mathbf{A}$ strongly μ-accelerates the convergence of a sequence $x \in c$ if the sequence $\mathbf{A}x$ converges strongly μ-faster than x.

Theorem 3. *For every $x \in c \setminus \varphi$ there exists a regular matrix $\mathbf{A}$, which strongly μ-accelerates the convergence of x.*

Proof By Corollary 2.1 of [1] every $x \in c \setminus \varphi$ has the limit speed $\lambda^* = (\lambda_n^*)$. We show that there exists a regular matrix $\mathbf{A}$ so that the limit speed of the sequence $(\mathbf{A}_n x)$ is higher than λ^*. As every $x = (x_n) \in c$ (with limit ς) can be presented in the form

$$x = x^0 + \varsigma e, \text{ where } x^0 = \left(x_n^0\right) \in c_0 \text{ and } e = (1, 1, ...), \tag{2.1}$$

where c_0 is the set of sequences, converging to zero, then we get

$$\lambda_n^* \, |x_n - \varsigma| = \lambda_n^* \left|x_n^0\right| = \mathcal{O}(1) \text{ or } \left|x_n^0\right| = \mathcal{O}\left(\frac{1}{\lambda_n^*}\right) \text{ and } \lambda_n^* \left|x_n^0\right| \neq o(1).$$

As the limit speed λ^* is a monotonically increasing unbounded sequence, then there exists a subsequence $\left(\lambda^*_{k_n}\right)$ of λ^* such that $\lambda^*_{k_n}/\lambda^*_n \longrightarrow \infty$. We define a matrix $\mathbf{A} = (a_{nk})$ by the equalities

$$a_{nk} = \begin{cases} 1 & k = k_n, \\ 0 & k \neq k_n. \end{cases}$$

With the help of Theorem 2.3.7 from [2] it is not difficult to check that the matrix $\mathbf{A}$ is regular. Now we have

$$\left|\mathbf{A}_n\, x^0\right| = \Big|\sum_k a_{nk} x^0_k\Big| = \left|x^0_{k_n}\right| = \mathcal{O}\Big(\frac{1}{\lambda^*_{k_n}}\Big)$$

or, equivalently,

$$\lambda^*_{k_n}\left|\mathbf{A}_n\, x^0\right| = \mathcal{O}(1).$$

Denoting $\mu = (\mu_n) = \left(\lambda^*_{k_n}\right)$, we get

$$\mu_n\left|\mathbf{A}_n\, x^0\right| = \mathcal{O}(1), \text{ where } \mu_n/\lambda^*_n \longrightarrow \infty.$$

Therefore $\mathbf{A}$ strongly μ-accelerates the convergence of x^0. As $\mathbf{A}_n e = 1$, then with the help of (2.1) we conclude

$$\mu_n\left|\mathbf{A}_n\, x - \varsigma\right| = \mu_n\left|\mathbf{A}_n\, x^0 + \varsigma \mathbf{A}_n\, e - \varsigma\right| = \mu_n\left|\mathbf{A}_n\, x^0\right|.$$

Consequently $\mathbf{A}$ strongly μ-accelerates also the convergence of x. ■

We note that the assertion of Theorem 3 does not hold for the concept of classical faster convergence. Indeed, it is not possible to accelerate the convergence of $x = (x_n) \in c \setminus \varphi$ by any regular matrix method if, for example, x is defined by the relation

$$x_n = \begin{cases} \dfrac{1}{n+1} & n = 2k, \\ 0 & n = 2k+1. \end{cases}$$

It follows from the proof of Theorem 3.2 of [1] that for every triangular regular matrix $\mathbf{A}$ there exists a convergent sequence x, which converges μ-faster than its $\mathbf{A}$-transform $\mathbf{A}x$. For strong μ-acceleration of convergence we can extract from the proof of Theorem 3.2 of [1] the following result.

Proposition 1. *If a triangular regular matrix* $\mathbf{A}$ *has a column with infinite number of non-zero elements, then there exists a sequence* x*, converging strongly* μ*-faster than its* $\mathbf{A}$*-transform* $\mathbf{A}x$*.*

As we see from Proposition 1, for some triangular regular methods $\mathbf{A}$ it is possible to choose a sequence x, converging strongly μ-faster than its $\mathbf{A}$-transform $\mathbf{A}x$, but it is not so for all triangular regular matrices.

Example 2. Let $\mathbf{A} = (a_{nk})$ be defined by the relation

$$a_{nk} = \begin{cases} \delta_{nk} & n = 2j, \\ \dfrac{1}{2} & n = 2j+1, \; k = n-1, n, \\ 0 & k < n-1, \end{cases}$$

where $j = 0, 1, \ldots$. Then for every convergent sequence $x = (x_k)$ we get

$$\mathbf{A}_n x = \begin{cases} x_n & n = 2j, \\ \dfrac{1}{2}(x_{n-1} + x_n) & n = 2j+1, \end{cases}$$

where $j = 0, 1, \ldots$. Now a sequence x can converge μ-faster than its **A**-transform $\mathbf{Ax}$ only in the case, if $x_{n-1}/x_n \neq \mathcal{O}(1)$. But never x can converge strongly μ-faster than its **A**-transform $\mathbf{A}x$.

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