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Several Theorems on λ–Summable Series*

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Abstract. We prove several propositions on λ–summable series by Cesàro method $(C, 1)$ or by weighted mean methods $\overline{N}$, which are also often called Riesz methods $P = (R, p_n)$.

Keywords: summability by weighted mean methods, λ-bounded series.

AMS Subject Classification: 40-XX.

1 Introduction

A sequence $x = \{\xi_n\}$ is called bounded with the rapidity $\lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow)$ if $\lambda_n (\xi_n - \xi) = O(1)$ with $\lim \xi_n = \xi$. A sequence $x = \{\xi_n\}$ is called λ-bounded by a matrix method A if Ax is λ-bounded. G. Kangro [2] proved Tauberian remainder theorem for the Riesz summability method A preserving λ-boundedness (by the supposition $Am^\lambda \subset m^\lambda$), where

$$m^\lambda = \{x | \ x = \{\xi_n\} \wedge \lim \xi_n = \xi \wedge \lambda_n(\xi_n - \xi) = O(1)\}.$$

I. Tammeraid [5] studied Tauberian remainder theorems for Cesàro and Hölder methods of summability. For example: if the sequences x and λ satisfy the conditions $n\lambda_n \Delta\xi_n = O(1)$, $x \in ((C, \alpha), m^\lambda)$ $(\alpha > 0)$ and

$$\frac{\lambda_n}{n+1} \sum_{k=0}^{n} \frac{1}{\lambda_k} = O(1), \tag{1.1}$$

then $x \in m^\lambda$.

If we inquire the condition (1.1) in the case $\lambda_n = (n + 1)^\alpha$, we get that $0 < \alpha < 1$. That means that the condition (1.1) of preserving λ-boundedness does not enable us to study these problems in the case $\lambda_n = (n + 1)^\alpha$ with $\alpha \geq 1$. Therefore we are interested in the ideas which gave us G. H. Hardy [1], F. Móricz and B. E. Rhoades [3, 4].

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2 Cesàro Means of Order One

We use two lemmas (see [1]).

Lemma 1. *If the series*

$$\sum_{k=0}^{\infty} a_k \tag{2.1}$$

is $(C,1)$*–summable, then the series*

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} \tag{2.2}$$

is convergent.

Lemma 2. *The necessary and sufficient condition that the series* (2.1) *should be summable* $(C,1)$ *to sum* A *is that*

$$\lim\left(\xi_n + (n+1)b_{n+1}\right) = A, \tag{2.3}$$

while

$$\xi_n = \sum_{k=0}^{n} a_k, \quad b_n = \sum_{k=n}^{\infty} \frac{a_k}{k+1}. \tag{2.4}$$

It is easy to control (see [1]) that the convergence of the series $\sum b_n$ to A is equivalent to the condition (2.3).

Proposition 1. *If* $0 < \mu_n \nearrow \infty$,

$$\mu_n = O(\mu_{n-1}) \tag{2.5}$$

and the series (2.1) *is* μ*–bounded by the method* $(C,1)$ *and*

$$\mu_n \sum_{k=n+1}^{\infty} \frac{1}{(k+2)(k+3)\mu_k} = O(1), \tag{2.6}$$

then the series (2.2) *is* μ*–bounded.*

Proof. Let the series (2.1) be $(C,1)$–summable to A. That means $\lim \sigma_n = A$, while

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} \xi_k, \quad \xi_n = (n+1)\sigma_n - n\sigma_{n-1}. \tag{2.7}$$

It is obvious (see [1]) that we may suppose without loss of generality that $A = 0$. As the series (2.1) is μ-bounded by $(C,1)$ and $A = 0$, then

$$\mu_n \sigma_n = O(1). \tag{2.8}$$

Using Lemma 1 we get that the series (2.2) is convergent. As

$$\sum_{k=0}^{n} \frac{a_k}{k+1} - \sum_{k=0}^{\infty} \frac{a_k}{k+1} = -\sum_{k=n+1}^{\infty} \frac{a_k}{k+1}$$

then using (2.4), (2.7) we get

$$\sum_{k=n+1}^{\infty} \frac{a_k}{k+1} = \sum_{k=n+1}^{\infty} \frac{\xi_k - \xi_{k-1}}{k+1} = -\frac{\xi_n}{n+2} + \sum_{k=n+1}^{\infty} \frac{\xi_k}{(k+1)(k+2)}$$
$$= \frac{n\sigma_{n-1}}{n+2} - \frac{(n+1)(n+4)\sigma_n}{(n+2)(n+3)} + 2\sum_{k=n+1}^{\infty} \frac{\sigma_k}{(k+2)(k+3)}.$$

Therefore using (2.5), (2.6) and (2.8) we have

$$\mu_n \sum_{k=n+1}^{\infty} \frac{a_k}{k+1} = \frac{n}{n+2}\frac{\mu_n}{\mu_{n-1}}\mu_{n-1}\sigma_{n-1} - \frac{(n+1)(n+4)}{(n+2)(n+3)}\mu_n\sigma_n$$
$$+ 2\mu_n \sum_{k=n+1}^{\infty} \frac{1}{(k+2)(k+3)\mu_k}\mu_k\sigma_k = O(1) + O(1) + O(1) = O(1).$$

So the assertion of Proposition 1 is valid. □

Proposition 2. *If* $0 < \lambda_n \nearrow \infty$,

$$\lambda_n = O(\lambda_{n-1}) \tag{2.9}$$

and the series (2.1) *is* λ*–bounded with* $\lambda = \{\lambda_n\}$ *by the method* $(C, 1)$ *and*

$$(n+1)\lambda_n \sum_{k=n+1}^{\infty} \frac{1}{(k+2)(k+3)\lambda_k} = O(1), \tag{2.10}$$

then the sequence

$$\{\xi_n + (n+1)b_{n+1}\}, \tag{2.11}$$

where the quantities ξ_n *and* b_n *are defined by* (2.4)*, is* λ*–bounded.*

Proof. Let the series (2.1) be $(C, 1)$–summable to A. Let $A = 0$. Using Lemma 2 we get that the sequence (2.11) is convergent to 0. So we have $\lambda_n\sigma_n = O(1)$. Using (2.4) and (2.7) we get

$$\xi_n + (n+1)b_{n+1} = \xi_n + (n+1)\sum_{k=n+1}^{\infty} \frac{\xi_k - \xi_{k-1}}{k+1},$$

$$\xi_n + (n+1)b_{n+1} = -\frac{n\sigma_{n-1}}{n+2} + \frac{2(n+1)\sigma_n}{(n+2)(n+3)} + 2(n+1)\sum_{k=n+1}^{\infty} \frac{\sigma_k}{(k+2)(k+3)}.$$

So we get

$$\lambda_n(\xi_n + (n+1)b_{n+1}) = -\frac{n}{n+2}\frac{\lambda_n}{\lambda_{n-1}}\lambda_{n-1}\sigma_{n-1} + \frac{2(n+1)}{(n+2)(n+3)}\lambda_n\sigma_n$$
$$+ 2(n+1)\lambda_n \sum_{k=n+1}^{\infty} \frac{1}{(k+2)(k+3)\lambda_k}\lambda_k\sigma_k.$$

Therefore using properties of σ_n, (2.9) and (2.10) we get

$$\lambda_n\left(\xi_n + (n+1)b_{n+1}\right) = O(1)\cdot O(1)\cdot O(1) + O(1)\cdot O(1) + O(1)\cdot O(1) = O(1).$$

Thus the sequence (2.11) is λ-bounded and the assertion of the Proposition 2 is valid. □

Proposition 3. *If* $0 < \lambda_n \nearrow \infty$,

$$\mu_n = (n+1)\lambda_n \tag{2.12}$$

and the series (2.1) *is* μ*–bounded by the method* $(C,1)$ *and the conditions* (2.9) *and* (2.10) *are satisfied, then the series* (2.1) *is* λ*–bounded.*

Proof. Let $A = 0$. Using (2.9) and (2.12) we get the condition (2.5) is satisfied. Using the Proposition 1 we get that the series (2.2) is μ–bounded. Using the Proposition 2 we get that the sequence (2.11) is λ-bounded. So we get

$$\lambda_n\xi_n + \lambda_n(n+1)\sum_{k=n+1}^{\infty}\frac{a_k}{(k+1)} = O(1).$$

As the series (2.2) is μ-bounded, we have

$$\lambda_n(n+1)\sum_{k=n+1}^{\infty}\frac{a_k}{k+1} = O(1).$$

So we get $\lambda_n\xi_n = O(1)$ and the assertion of the Proposition 3 is valid. □

3 Weighted Means

F. Móricz, B. E. Rhoades [3] and [4] used Hardy's idea for an equivalent reformulation of summability by weighted mean methods. Let $\{p_k\}$ be a fixed sequence of positive numbers and $P_n = \sum_{k=0}^{n} p_k$. A series (2.1) is said to be summable by the weighted mean method $\overline{N}$ (often called as Riesz method $P = (R, p_n)$) if the sequence $\{\eta_n\}$ defined by

$$\eta_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k\xi_k, \tag{3.1}$$

where ξ_k is defined by (2.4), converges to a finite limit as $n \to \infty$. We use a (see [3] and [4])

Lemma 3. *Let* $\overline{N}$ *be the weighted mean method determined by* $\{p_n\}$ *satisfying the conditions*

$$p_n \ge a > 0 \quad (n = 0,1,2,\ldots), \quad p_{n+1}/p_n = O(1), \tag{3.2}$$

$$\frac{p_{n+1}P_n}{p_n} \nearrow, \quad P_n \nearrow \infty. \tag{3.3}$$

If the series (2.1) *is* $\overline{N}$*–summable to a finite number* A, *then the series*

$$\sum_{n=0}^{\infty} b_n \tag{3.4}$$

converges to A, *while*

$$b_n = p_n \sum_{k=n}^{\infty} \frac{a_k}{P_k}. \tag{3.5}$$

Let

$$\zeta_n = \xi_n + \frac{P_n}{p_{n+1}} b_{n+1}, \tag{3.6}$$

while the quantity ξ_n is defined by (2.4).

Remark 1. The convergence of the series (3.4) to A is equivalent (see [4]) to the limit relation

$$\lim \zeta_n = A.$$

Proposition 4. *If* $0 < \lambda_n \nearrow \infty$ *and the conditions* (2.9), (3.2), (3.3),

$$p_{n+1}^2 P_{n+2} - p_n p_{n+2} P_n \geq 0, \tag{3.7}$$

$$\lambda_n P_n \sum_{k=n+1}^{\infty} \frac{p_{k+1}^2 P_{k+2} - p_k p_{k+2} P_k}{\lambda_k p_k p_{k+1} P_{k+1} P_{k+2}} = O(1) \tag{3.8}$$

are satisfied and the series (2.1) *is* λ*–bounded by the method* $\overline{N}$, *then the sequence* $\{\zeta_n\}$ *is* λ*–bounded.*

Proof. Let the series (2.1) be $\overline{N}$–summable to A. That means $\lim \eta_n = A$. Using Lemma 3 and Remark 1 we get $\lim \zeta_n = A$. It is easy to prove (see [4]) that we may suppose without loss of generality that $A = 0$. So we have $\lambda_n \eta_n = O\left(1\right)$. As by (3.1) we have

$$\xi_n = \left(P_n \eta_n - P_{n-1} \eta_{n-1}\right) / p_n, \tag{3.9}$$

then using (3.6), (3.5) and (3.9) we get

$$\begin{aligned} \zeta_n = &- \frac{p_{n+1} P_{n-1}}{p_n P_{n+1}} \eta_{n-1} + \frac{p_{n+1}^2 P_n P_{n+2} - p_n p_{n+2} P_n^2}{p_n p_{n+1} P_{n+1} P_{n+2}} \eta_n \\ &+ P_n \sum_{k=n+1}^{\infty} \frac{p_{k+1}^2 P_{k+2} - p_k p_{k+2} P_k}{p_k p_{k+1} P_{k+1} P_{k+2}} \eta_k. \end{aligned}$$

As $\lambda_k \eta_k = O(1)$, then using (2.9), (3.2), (3.3), (3.7) and (3.8) we get

$$\begin{aligned} \lambda_n \zeta_n = &- \frac{p_{n+1} P_{n-1}}{p_n P_{n+1}} \frac{\lambda_n}{\lambda_{n-1}} \lambda_{n-1} \eta_{n-1} + \frac{p_{n+1}^2 P_n P_{n+2} - p_n p_{n+2} P_n^2}{p_n p_{n+1} P_{n+1} P_{n+2}} \lambda_n \eta_n \\ &+ \lambda_n P_n \sum_{k=n+1}^{\infty} \frac{p_{k+1}^2 P_{k+2} - p_k p_{k+2} P_k}{\lambda_k p_k p_{k+1} P_{k+1} P_{k+2}} \lambda_k \eta_k \\ &= O(1)O(1)O(1) + O(1)O(1) + O(1) = O(1). \end{aligned}$$

□

Proposition 5. *If* $0 < \lambda_n \nearrow \infty$, $\mu_n = P_n \lambda_n$, $\gamma_n = P_n b_{n+1}/p_{n+1}$ *and the series* (2.1) *is* μ*–bounded by the method* $\overline{N}$ *and the conditions* (2.9), (3.2), (3.3), (3.7) *and* (3.8) *are satisfied, then the sequence* $\{\gamma_n\}$ *is* λ*–bounded.*

Proof. Let $A = 0$. Then we have $\mu_n \eta_n = O(1)$. As the series (2.1) is μ-bounded then this series is also λ-bounded. So we get

$$\lambda_n \gamma_n = \lambda_n P_n \sum_{k=n+1}^{\infty} \frac{p_{k+1}^2 P_{k+2} - p_k p_{k+2} P_k}{\lambda_k p_k p_{k+1} P_{k+1} P_{k+2}} \lambda_k \eta_k + \frac{\lambda_n}{\lambda_{n-1}} \frac{P_{n-1}}{p_n P_{n+1}} \lambda_{n-1} \eta_{n-1}$$
$$- \left(\frac{p_{n+2} P_n}{p_{n+1} P_{n+1} P_{n+2}} + \frac{P_n}{p_n P_{n+1}} \right) \mu_n \eta_n = O(1).$$

□

Proposition 6. *If the conditions of Propositions* 4 *and* 5 *are satisfied, then the series* (2.1) *is* λ*–bounded.*

Proof. Let $A = 0$. Using (3.6) we get

$$\lambda_n \zeta_n = \lambda_n \xi_n + \lambda_n \gamma_n.$$

As $\lambda_n \zeta_n = O(1)$ and $\lambda_n \gamma_n = O(1)$ then $\lambda_n \xi_n = O(1)$. □

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