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Involving Fuzzy Orders for Multi-Objective Linear Programming

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Abstract. This paper presents a solution approach for multi-objective linear programming problem. We propose to involve fuzzy order relations to describe the objective functions where in "classical" fuzzy approach the membership functions which illustrate how far the concrete point is from the solution of individual problem are studied. Further the global fuzzy order relation is constructed by aggregating the individual fuzzy order relations. Thus the global fuzzy relation contains the information about all objective functions and in the last step we find a maximum in the set of constrains with respect to the global fuzzy order relation. We illustrate this approach by an example.

Keywords: multi-objective linear programming, fuzzy order relation, aggregation of fuzzy relations.

AMS Subject Classification: 90C29; 03E72; 06A06.

1 Introduction

In the paper we work in the field of multi-objective (or Multiple Objective) linear programming (MOLP), which is an important tool for solving real-life optimization problems such as production planning, logistics, environment management, banking/finance planning etc. Our investigations are based on the fuzzy approach [16] where the membership functions are involved to prescribe how far the concrete point is from the solution of an individual problem. In our paper we propose to use fuzzy order relations [1, 15] instead of the membership functions described above. Further we describe the solution approach and investigate examples. Let us now focus on the problem formulation and the scheme description.

MOLP problem can be represented as follows:

$$\max Z, \quad \text{where } Z = (z_1, \ldots, z_k) \text{ is a vector of objectives,}$$
$$z_i = \sum_{j=1}^{n} c_{ij} x_j, \quad \text{where } i = 1, \ldots, k,$$
$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m. \tag{1.1}$$

So we must find a vector $\mathbf{x}^o = (x_1^o, \ldots, x_n^o)$ which maximizes k objective functions of n variables, and with m constraints. Let D denote a feasible region of the problem (1.1). For the sake of brevity further we denote vectors in bold, e.g., $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y}^* = (y_1^*, \ldots, y_n^*)$.

In problem (1.1), all objective functions can hardly reach their optima at the same time subject to the given constraints since usually the objective functions conflict with one another. Thus Pareto optimal solution (efficient solution) and optimal compromise solution are introduced:

Definition 1. [17] $\mathbf{x}^*$ is called Pareto optimal solution if and only if there does not exist another $\mathbf{x} \in D$ such that $z_i(\mathbf{x}^*) \le z_i(\mathbf{x})$ for all i and $z_j(\mathbf{x}^*) \ne z_j(\mathbf{x})$ for at least one j.

Definition 2. [17] An optimal compromise solution of a vector-maximum problem is a solution $\mathbf{x} \in D$ which is preferred by the decision maker to all other solutions, taking into consideration all criteria contained in the vector-valued objective function. It is generally accepted, that an optimal compromise solution has to be a Pareto optimal solution. Further we will call optimal compromise solution simply optimal solution.

Thus our main aim is to determine the optimal compromise solution. The fuzzy approach for solving MOLP proposed by Zimmermann [16] has given an effective way of measuring the satisfaction degree for MOLP. The idea is to identify the membership functions prescribing the fuzzy goals (solutions of individual problem) for the objective functions z_i, $i = 1, \ldots, k$. The following linear function is an example of a membership function:

$$\mu_i(\mathbf{x}) = \begin{cases} 0, & \text{if } z_i(\mathbf{x}) < z_i^{min}, \\ \dfrac{z_i(\mathbf{x}) - z_i^{min}}{z_i^{max} - z_i^{min}}, & z_i^{min} \le z_i(\mathbf{x}) \le z_i^{max}, \\ 1, & z_i(\mathbf{x}) > z_i^{max}, \end{cases}$$

where z_i^{max} is the solution of individual problem

$$\max z_i, \quad \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m$$

and z_i^{min} is the solution of individual problem

$$\min z_i, \quad \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m.$$

Usually the membership functions μ_i are linear functions and it is argued by the "facilitation computation for obtaining solutions". Further in the "classical" fuzzy approach membership functions μ_i are aggregated. The main subject which is discussed in the large part of papers is the choice of an aggregation function.

In our paper we propose a completely different approach although we still use the fuzzy environment. We initiate involving of fuzzy orders to solve the problem. To justify the choice of fuzzy order let us first observe the classical linear programming problem when we should maximize the unique function $z = \sum_{j=1}^{n} c_j x_j$ where the vectors $(x_1, \ldots, x_n)$ belong to the set

$$D\colon \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m.$$

In this case we can involve the relation $\preceq$:

$$\mathbf{x} \preceq \mathbf{y} \quad \Leftrightarrow \quad z(\mathbf{x}) \leq z(\mathbf{y})$$

which is obviously a crisp linear order with respect to the crisp equivalence relation

$$\mathbf{x} \doteq \mathbf{y} \quad \Leftrightarrow \quad z(\mathbf{x}) = z(\mathbf{y}).$$

Thus we can reformulate the problem in the following way: $MAX(D, \preceq)$. That is we should find a maximum in the set D which is ordered by the linear order $\preceq$. We use this idea to solve the multi-objective linear programming problem. Since we have more than one objective function we should involve order relation for each objective function and they should be obviously fuzzy order relations to overcome the conflict of all objective functions. Further we aggregate fuzzy order relations to get one fuzzy order relation which include the information about all objective functions and in the last step we should find a maximum in the set D with respect to the aggregated fuzzy order relation. Thus the scheme of solution is as follows:

1. We define fuzzy order relations P_i which generalize the following crisp order relations:

   $$\mathbf{x} \preceq_i \mathbf{y} \quad \Leftrightarrow \quad z_i(\mathbf{x}) \leq z_i(\mathbf{y}), \quad i = 1, \ldots, k.$$

   Thus each fuzzy order relation describes corresponding objective function z_i.

2. We aggregate fuzzy orders using an aggregation function A which preserves the properties of initial fuzzy orders:

   $$P(\mathbf{x}, \mathbf{y}) = A\big(P_1(\mathbf{x}, \mathbf{y}), \ldots, P_k(\mathbf{x}, \mathbf{y})\big).$$

   Thus the aggregated fuzzy order relation P provides the information about all objective functions.

3. We find a maximum in the set D with respect to the aggregated fuzzy order relation P.

In our work we exactly realize the above described scheme. As we have seen above, solving the classical linear programming problem with one objective function there is naturally arisen crisp linear order, which could be naturally generalized to fuzzy linear order solving multi-objective problem. Thus if we use fuzzy approach proposed by Zimmermann [16] and generalized by many others authors (see e.g. [6, 11, 12]) we do not take into account the information about these orders (which are reflective, transitive and antisymmetric relations), so this information is lost. Thus one of the advantages of our approach is that we take into account this information, and even more aggregating these fuzzy orders we use the aggregation function which preserve the properties of fuzzy orders. The other advantage is that in our approach we explain the "shape" of fuzzy order relation and choice of aggregation function (this is caused by the fuzzy environment (or t-norm) in which we are working). Moreover, in our approach we can naturally use compensatory aggregation functions and even more we can use weights (see last section) to show the preference of objective functions.

As we wrote in Definition 2, an optimal compromise solution has to be a Pareto optimal solution. Although the "min" operator method, proposed by Zimmermann [16] has been proven to have several nice properties, the solution generated by this approach does not guarantee Pareto-optimality. As we will see later, in our approach we have found the properties which guarantee Pareto-optimality even regardless of the uniqueness of the optimal solution (see Theorem 6).

The paper is structured in the following way: Section 2 contains some known facts about fuzzy logic important for the further understanding of the material; we propose the general information about fuzzy relations and build the essential fuzzy relations for the realization of our scheme in Section 3; we study the aggregation of fuzzy relations in Section 4; We propose the solution approach in Section 5; we observe the numerical example in Section 6 and we conclude our paper by Section 7.

2 Preliminaries

When we solve MOLP problem using a fuzzy approach it is worth to work in fuzzy logic where the truth values are from the unit interval with 1 being the absolute truth and 0 being the absolute falsity. For example for the statement that $x \leq y$ we do not say that it is true or false, but we give the degree to which the statement is true which is a number from the unit interval. For the brief introduction to fuzzy set theory and fuzzy logic see [14], for more detailed information see [7]. We start with the definition of a t-norm which represents a generalized conjunction in fuzzy logic:

Definition 3. [9] A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i.e. a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:

- $T(x, y) = T(y, x)$ (commutativity);
- $T(x, T(y, z)) = T(T(x, y), z)$ (associativity);
- $T(x, y) \leq T(x, z)$ whenever $y \leq z$ (monotonicity);
- $T(x, 1) = x$ (boundary condition).

Some of often used t-norms are mentioned below:

- $T_M(x, y) = \min(x, y)$ minimum t-norm;
- $T_P(x, y) = x \cdot y$ product t-norm;
- $T_L(x, y) = \max(x + y - 1, 0)$ Łukasiewicz t-norm.

A t-norm T is called Archimedean if and only if, for all pairs $(x, y) \in (0, 1)^2$, there is $n \in \mathbb{N}$ such that $x_T^{(n)} < y$. Product and Łukasiewicz t-norms are Archimedean while minimum t-norm is not.

We proceed with one powerful tool for the construction of t-norms involving only one-place real function (additive generator) and addition. Furthermore, we use the same tool for constructing fuzzy equivalence.

Definition 4. [9] Let $f : [a, b] \to [c, d]$ be a monotone function, where $[a, b]$ and $[c, d]$ are closed subintervals of the extended real line $[-\infty, \infty]$. The pseudo-inverse $f^{(-1)} : [c, d] \to [a, b]$ of f is defined by

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{if } f(a) < f(b), \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{if } f(a) > f(b), \\ a & \text{if } f(a) = f(b). \end{cases}$$

Definition 5. [9] An additive generator $t : [0, 1] \to [0, \infty]$ of a t-norm T is a strictly decreasing function which is also right-continuous in 0 and satisfies $t(1) = 0$, such that for all $(x, y) \in [0, 1]^2$ we have

$$t(x) + t(y) \in Ran(t) \cup \big[t(0), \infty\big], \quad T(x, y) = t^{(-1)}\big(t(x) + t(y)\big).$$

3 Fuzzy Order Relations

We continue with an overview of basic definitions and results on fuzzy relations. Definitions of a fuzzy order relation and a fuzzy equivalence relation were first introduced by L.A. Zadeh in 1971 [15] under the names of fuzzy ordering and similarity relation. Fifteen years later U. Höhle and N. Blanchard in their paper [8] proposed to involve fuzzy equivalence relation (L-valued equality) in a definition of a fuzzy order (partial ordering). In our paper we use more resent results on fuzzy order defined with respect to the fuzzy equivalence relation (studied in [1]).

Definition 6. A fuzzy binary relation R on a set X is a mapping $R : X \times X \to [0, 1]$.

Definition 7 [see e.g. [1]]. A fuzzy binary relation E on a set X is called a fuzzy equivalence relation with respect to a t-norm T (or T-equivalence), if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

1. $E(x,x) = 1$ reflexivity;
2. $E(x,y) = E(y,x)$ symmetry;
3. $T(E(x,y), E(y,z)) \leq E(x,z)$ T-transitivity.

The following result establishes principles of construction of fuzzy equivalence relations using pseudo-metrics.

Theorem 1. [4] *Let T be a continuous Archimedean t-norm with an additive generator t. For any pseudo-metric d, the mapping*

$$E_d(x,y) = t^{(-1)}\bigl(\min\bigl(d(x,y), t(0)\bigr)\bigr)$$

is a T-equivalence.

Example 1. Let us consider the set of real numbers $X = \mathbb{R}$ and metric $d(x,y) = |x-y|$ on it. Taking into account that $t_L(x) = 1 - x$ is an additive generator of T_L (Łukasiewicz t-norm) and that $t_P(x) = -\ln(x)$ is an additive generator of T_P (product t-norm), we obtain two fuzzy equivalence relations:

$$E_L(x,y) = \max\bigl(1 - |x-y|, 0\bigr); \quad E_P(x,y) = e^{-|x-y|}.$$

Definition 8 [see e.g. [1]]. A fuzzy binary relation L on a set X is called fuzzy order relation with respect to a t-norm T and a T-equivalence E (or T-E-order), if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

1. $L(x,y) \geq E(x,y)$ E-reflexivity;
2. $T(L(x,y), L(y,z)) \leq L(x,z)$ T-transitivity;
3. $T(L(x,y), L(y,x)) \leq E(x,y)$ T-E-antisymmetry.

A fuzzy order relation L is called strongly linear if and only if $\forall x, y \in X$: $\max(L(x,y), L(y,x)) = 1$.

The following theorem states that strongly linear fuzzy order relations are uniquely characterized as fuzzifications of crisp linear orders. Preliminarily let us recall the definition of compatibleness:

Definition 9. [1] Let $\preceq$ be a crisp order on X and let E be a fuzzy equivalence relation on X. E is called compatible with $\preceq$ if and only if the following implication holds for all $x, y, z \in X$: $x \preceq y \preceq z \Rightarrow (E(x,z) \leq E(y,z)$ and $E(x,z) \leq E(x,y))$.

Theorem 2. [1] *Let L be a binary fuzzy relation on X and let E be a T-equivalence on X. Then the following two statements are equivalent*:

1. *L is a strongly linear T-E-order on X.*

2. *There exists a linear order $\preceq$ the relation E is compatible with, such that L can be represented as follows*:

$$L(x,y) = \begin{cases} 1, & \textit{if } x \preceq y, \\ E(x,y), & \textit{otherwise.} \end{cases}$$

This theorem shows that if we have a set X, a linear order $\preceq$ on it and a T-equivalence on X which is compatible with $\preceq$, then we can build a fuzzy linear order L as it was shown above.

Let us now come back to the realization of our scheme. Our aim now is to involve fuzzy orders P_i which contain the information about objective functions z_i. Since we define fuzzy order relations it is necessary to define fuzzy equivalence relations first. To define the fuzzy equivalence relations we use the construction proposed in the Theorem 1 where the relation is constructed on the base of a pseudo-metric. It is worth to mention that this approach is widely used in the literature for practical applications (see e.g. [3]).

Thus we build the following pseudo-metrics on the set D:

$$d_i(\mathbf{x},\mathbf{y}) = \frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}.$$

Thus defined d_i are indeed pseudo-metrics and applying the Theorem 1 we can build a T-equivalence relation:

$$E_i(\mathbf{x},\mathbf{y}) = t^{(-1)}\left(\min\left(\frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}, t(0)\right)\right), \tag{3.1}$$

where t is an additive generator of a continuous Archimedean t-norm T.

Hence we should first choose a t-norm which plays a role of a generalized conjunction and further construct T-equivalences using a correspondent additive generator t.

Example 2.

1.
$$E_i(\mathbf{x},\mathbf{y}) = 1 - \frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}} \tag{3.2}$$

are fuzzy T_L-equivalence relations.

2.
$$E_i(\mathbf{x},\mathbf{y}) = e^{-\frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}} \tag{3.3}$$

are fuzzy T_P-equivalence relations.

Remark 1. Although the above defined pseudo-metrics are quite natural, other metrics can be also used. For example the following pseudo-metrics can be chosen:

$$d_i(\mathbf{x},\mathbf{y}) = C_i \cdot \frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}},$$

where C_i is a real number greater than 0.

In this case $E_i(\mathbf{x},\mathbf{y}) = \max(1 - C_i \cdot \frac{|z_i(\mathbf{x})-z_i(\mathbf{y})|}{z_i^{max}-z_i^{min}}, 0)$ are fuzzy T_L-equivalence relations and $E_i(\mathbf{x},\mathbf{y}) = e^{-C_i \frac{|z_i(\mathbf{x})-z_i(\mathbf{y})|}{z_i^{max}-z_i^{min}}}$ are fuzzy T_P-equivalence relations.

Further we build fuzzy order relations applying Theorem 2. Namely we construct T-E_i-orders where T is a chosen t-norm and E_i is a constructed fuzzy equivalence relation. To apply Theorem 2 we should also fix crisp order relations and in our case they are linear orders $\preceq_i$ on the set D:

$$\mathbf{x} \preceq_i \mathbf{y} \quad \Leftrightarrow \quad z_i(\mathbf{x}) \le z_i(\mathbf{y}).$$

Let us show that fuzzy equivalence relation (3.1) is compatible with linear order $\preceq_i$: $\mathbf{x} \preceq_i \mathbf{y} \preceq_i \mathbf{z} \Rightarrow (E_i(\mathbf{x},\mathbf{z}) \le E_i(\mathbf{y},\mathbf{z})$ and $E_i(\mathbf{x},\mathbf{z}) \le E_i(\mathbf{x},\mathbf{y}))$.

If $\mathbf{x} \preceq_i \mathbf{y} \preceq_i \mathbf{z}$ then $z_i(\mathbf{x}) \le z_i(\mathbf{y}) \le z_i(\mathbf{z})$ and hence $|z_i(\mathbf{x}) - z_i(\mathbf{y})| \le |z_i(\mathbf{x}) - z_i(\mathbf{z})|$. Furthermore

$$\min\left(\frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}, t(0)\right) \le \min\left(\frac{|z_i(\mathbf{x}) - z_i(\mathbf{z})|}{z_i^{max} - z_i^{min}}, t(0)\right).$$

Hence by strictly decreasing monotonicity of $t^{(-1)}$ we get: $E_i(\mathbf{x},\mathbf{z}) \le E_i(\mathbf{x},\mathbf{y})$. The same considerations are valid to show that $E_i(\mathbf{x},\mathbf{z}) \le E_i(\mathbf{y},\mathbf{z})$.

Hence the following functions:

$$P_i(\mathbf{x},\mathbf{y}) = \begin{cases} 1, & \text{if } \mathbf{x} \preceq_i \mathbf{y} \\ E_i(\mathbf{x},\mathbf{y}), & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } z_i(\mathbf{x}) \le z_i(\mathbf{y}), \\ E_i(\mathbf{x},\mathbf{y}), & \text{otherwise.} \end{cases} \tag{3.4}$$

are T-E_i-orders, where E_i are defined by Equation (3.1).

Example 3.

1. $$P_i(\mathbf{x},\mathbf{y}) = \begin{cases} 1, & \text{if } z_i(\mathbf{x}) \le z_i(\mathbf{y}), \\ 1 - \frac{|z_i(\mathbf{x})-z_i(\mathbf{y})|}{z_i^{max}-z_i^{min}}, & \text{otherwise} \end{cases}$$

   are fuzzy order relations with respect to t-norm T_L and T_L-equivalence E_i defined by Equation (3.2).

2. $$P_i(\mathbf{x},\mathbf{y}) = \begin{cases} 1, & \text{if } z_i(\mathbf{x}) \le z_i(\mathbf{y}), \\ e^{-\frac{|z_i(\mathbf{x})-z_i(\mathbf{y})|}{z_i^{max}-z_i^{min}}}, & \text{otherwise} \end{cases}$$

   are fuzzy order relations with respect to t-norm T_P and T_P-equivalence E_i defined by Equation (3.3).

The fuzzy order relations are constructed and we come to the next step where we aggregate corresponding relations.

4 Aggregation of Fuzzy Order Relations

The idea of the following section is that we have to fuse the information about all fuzzy order relations P_i and get a global fuzzy order relation P which includes the information about all fuzzy order relations P_i and thereby also the information about all objective functions z_i. Let us introduce the following mapping $A : [0,1]^k \to [0,1]$ which aggregates fuzzy order relations:

$$P(\mathbf{x}, \mathbf{y}) = A\big(P_1(\mathbf{x}, \mathbf{y}), \dots, P_k(\mathbf{x}, \mathbf{y})\big).$$

It is natural to require from A at least the following properties:

1. If $P_i(\mathbf{x}, \mathbf{y}) = 1$ for all i (that is $\mathbf{x} \preceq_i \mathbf{y}$) the global degree should be also 1. In other words: $A(1, \dots, 1) = 1$.

2. If $\mathbf{x} \preceq_i \mathbf{y}$ does not entirely fulfilled for every i, then the global degree of fulfillment should be 0, too: $A(0, \dots, 0) = 0$.

3. If one degree $P_i(\mathbf{x}, \mathbf{y})$ increases while the others are kept constant, the overall degree must not decrease, i.e. A should be non-increasing in each component.

That is exactly the definition of aggregation function:

Definition 10. [5] An aggregation function is a mapping $A : [0,1]^k \to [0,1]$ which fulfills the following properties:

- $A(x_1, \dots, x_k) \leq A(y_1, \dots, y_k)$ whenever $x_i \leq y_i$ for all $i \in \{1, \dots, k\}$ (monotonicity);
- $A(0, \dots, 0) = 0$ and $A(1, \dots, 1) = 1$ (boundary conditions).

For more information about aggregation functions or aggregation operators see [5] and [10]. It is also natural to require that the global fuzzy relation should fulfill the same properties as the individual fuzzy relations.

Due to the fact that the fuzzy order relations are based on the equivalence relations let us first focus on the aggregation of fuzzy equivalence relations E_i. The preservation of reflexivity is rather clear because of the boundary conditions of aggregation function. Preservation of symmetry is also obvious. The more interesting and complex question is about preservation of T-transitivity. Here we use the results about the preservation of T-transitivity studied in [13], where it is shown that preservation of T-transitivity is equivalent to the dominance of the t-norm T by the aggregation operator (or function) A.

Definition 11. [13] Consider an n-argument aggregation function A:$[0,1]^n \to [0,1]$ and a t-norm T. We say that A dominates T if for all $x_i \in [0,1]$ with $i \in \{1, \dots, n\}$ and $y_i \in [0,1]$ with $i \in \{1, \dots, n\}$ the following property holds:

$$T\big(A(x_1, \dots, x_n), A(y_1, \dots, y_n)\big) \leq A\big(T(x_1, y_1), \dots, T(x_n, y_n)\big).$$

Theorem 3. [13] *Let $|X| > 3$ and let T be a t-norm. An aggregation function A preserves T-transitivity of fuzzy relations on X if and only if A belongs to the class of aggregation functions which dominate T.*

Corollary 1. Let $|X| > 3$ and let T be a t-norm. If E_i for all $i \in \{1, \dots, n\}$ are fuzzy equivalence relations (T-equivalences) then

$$E(x, y) = A\big(E_1(x, y), \dots, E_n(x, y)\big)$$

is also a T-equivalence relation if A belongs to the class of aggregation functions which dominate T.

We continue with the aggregation of fuzzy order relations. The next theorem is a straightforward generalization of Theorem 6.1 of [2].

Theorem 4. *Let $|X| > 3$ and let T be a t-norm. If E_i for all $i \in \{1,\ldots,n\}$ are fuzzy equivalence relations (T-equivalences); P_i for all $i \in \{1,\ldots,n\}$ are fuzzy order relations (T-E_i-orders) then $P(x,y) = A(P_1(x,y),\ldots,P_n(x,y))$ is T-E-order relation if A belongs to the class of aggregation functions which dominate T and $E(x,y) = A(E_1(x,y),\ldots,E_n(x,y))$.*

Proof.

1. Since all P_i are E_i-reflexive ($E_i(x,y) \le P_i(x,y)$)
$$A\big(E_1(x,y),\ldots,E_n(x,y)\big) \le A\big(P_1(x,y),\ldots,P_n(x,y)\big)$$
because of the monotonicity of the function A. Thus P is an E-reflexive fuzzy relation.
2. T-transitivity holds because of Theorem 3.
3. It remains to prove that $T(P(x,y),P(y,x)) \le E(x,y)$:
$$\begin{aligned}&T\big(P(x,y),P(y,x)\big)\\ &\quad = T\big(A\big(P_1(x,y),\ldots,P_n(x,y)\big),A\big(P_1(y,x),\ldots,P_n(y,x)\big)\big)\\ &\quad \le A\big(T\big(P_1(x,y),P_1(y,x)\big),\ldots,T\big(P_n(x,y),P_n(y,x)\big)\big)\end{aligned}$$
because of the dominance of T by A. Further
$$\begin{aligned}&A\big(T\big(P_1(x,y),P_1(y,x)\big),\ldots,T\big(P_n(x,y),P_n(y,x)\big)\big)\\ &\quad \le A\big(E_1(x,y),\ldots,E_n(x,y)\big)\end{aligned}$$
since A is a monotone function and $T(P_i(x,y),P_i(y,x)) \le E_i(x,y)$. Thus we have proven the required inequality. □

It was important to find conditions for an aggregation function which guarantee the preservation of the properties of fuzzy order relations in the aggregation process. Let us show the importance of the requirement that aggregated fuzzy relation of fuzzy orders must be also a fuzzy order by the example of preservation of transitivity:

If $z_i(\mathbf{x}) \le z_i(\mathbf{y})$ and $z_i(\mathbf{y}) \le z_i(\mathbf{z})$ for all i it is natural that the element $\mathbf{z}$ is more preferable for us than the element $\mathbf{x}$ in a global sense what is exactly guaranteed by the preservation of transitivity.

In the next two examples we observe the aggregation function which dominates Łukasiewicz and product t-norms:

Example 4. For any $k > 2$ and any $p = (p_1,\ldots,p_k)$ with $\sum_{i=1}^{k} p_i \ge 1$ and $p_i \in [0,\infty]$ k-ary aggregation function
$$A_p(x_1,\ldots,x_k) = \max\bigg(\sum_{i=1}^{k} x_i p_i + 1 - \sum_{i=1}^{k} p_i, 0\bigg)$$
dominates Łukasiewicz t-norm T_L.

Example 5. For any $k > 2$ and any $p = (p_1,\ldots,p_k)$ with $\sum_{i=1}^{k} p_i \ge 1$ and $p_i \in [0,\infty]$ k-ary aggregation function $A_p(x_1,\ldots,x_k) = \prod_{i=1}^{k} x_i^{p_i}$ dominates product t-norm T_P.

5 Solution Approach

Further the multi-objective linear programming problem comes to the following problem:

$$\max_{\mathbf{y}} \min_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}) \tag{P}$$

Intuitively this means that we find for each $\mathbf{y} \in D$ the value $\min\limits_{\mathbf{x}} P(\mathbf{x}, \mathbf{y})$, that is we find the degree to which $\mathbf{y}$ is greater (or better) than every $\mathbf{x} \in D$. In other words we find the degree to which $\mathbf{y}$ is a maximal element in the set D and later on we find $\mathbf{y}$ to which this satisfaction degree is the greatest.

Theorem 5. *An optimal solution* $\mathbf{y}$ *to the problem* (P) *is a Pareto optimal solution if it is the unique optimal solution.*

Proof. If $\mathbf{y}$ is not a Pareto optimal solution then there exists another $\tilde{\mathbf{y}} \in D$ such that $z_i(\mathbf{y}) \leq z_i(\tilde{\mathbf{y}})$ for all i and $z_j(\mathbf{y}) \neq z_j(\tilde{\mathbf{y}})$ for at least one j. Let us now compare $P_i(\mathbf{x}, \mathbf{y})$ and $P_i(\mathbf{x}, \tilde{\mathbf{y}})$. Further we distinguish between the following three cases:

1. If $z_i(\mathbf{y}) \leq z_i(\tilde{\mathbf{y}}) \leq z_i(\mathbf{x})$ or $z_i(\mathbf{y}) < z_i(\tilde{\mathbf{y}}) \leq z_i(\mathbf{x})$ then $|z_i(\mathbf{x}) - z_i(\tilde{\mathbf{y}})| \leq |z_i(\mathbf{x}) - z_i(\mathbf{y})|$. Furthermore
$$\min\left(\frac{|z_i(\mathbf{x}) - z_i(\tilde{\mathbf{y}})|}{z_i^{max} - z_i^{min}}, t(0)\right) \leq \min\left(\frac{|z_i(\mathbf{x}) - z_i(\mathbf{y})|}{z_i^{max} - z_i^{min}}, t(0)\right).$$
Further since $t^{(-1)}$ is strictly decreasing we get: $E_i(\mathbf{x}, \mathbf{y}) \leq E_i(\mathbf{x}, \tilde{\mathbf{y}})$ and thus $P_i(\mathbf{x}, \mathbf{y}) \leq P_i(\mathbf{x}, \tilde{\mathbf{y}})$.
2. If $z_i(\mathbf{x}) \leq z_i(\mathbf{y}) \leq z_i(\tilde{\mathbf{y}})$ or $z_i(\mathbf{x}) \leq z_i(\mathbf{y}) < z_i(\tilde{\mathbf{y}})$ then $P_i(\mathbf{x}, \mathbf{y}) = P_i(\mathbf{x}, \tilde{\mathbf{y}}) = 1$ since $z_i(\mathbf{x}) \leq z_i(\mathbf{y})$ and $z_i(\mathbf{x}) < z_i(\tilde{\mathbf{y}})$ (or $z_i(\mathbf{x}) \leq z_i(\tilde{\mathbf{y}})$).
3. If $z_i(\mathbf{y}) < z_i(\tilde{\mathbf{y}})$ then there could be also the following situation: $z_i(\mathbf{y}) < z_i(\mathbf{x}) < z_i(\tilde{\mathbf{y}})$. Then $P_i(\mathbf{x}, \mathbf{y}) \leq P_i(\mathbf{x}, \tilde{\mathbf{y}})$ since $P_i(\mathbf{x}, \tilde{\mathbf{y}}) = 1$.

Thus for all $\mathbf{x} \in D$

$$A\big(P_1(\mathbf{x}, \mathbf{y}), P_2(\mathbf{x}, \mathbf{y}), \ldots, P_k(\mathbf{x}, \mathbf{y})\big) \leq A\big(P_1(\mathbf{x}, \tilde{\mathbf{y}}), P_2(\mathbf{x}, \tilde{\mathbf{y}}), \ldots, P_k(\mathbf{x}, \tilde{\mathbf{y}})\big).$$

Hence $\min\limits_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}) \leq \min\limits_{\mathbf{x}} P(\mathbf{x}, \tilde{\mathbf{y}})$. This contradicts the fact that $\mathbf{y}$ is the unique optimal solution to the problem. □

We can also prove the above theorem without demanding the "uniqueness of the optimal solution" but in this case we should require some specific properties:

Theorem 6. *An optimal solution* $\mathbf{y}$ *to the problem* (P) *is a Pareto optimal solution if* $z_i(\mathbf{x}) > z_i(\mathbf{y}) \Rightarrow P_i(\mathbf{x}, \mathbf{y}) < 1$*,* A *is a strictly monotone function and set* D *is linearly connected.*

The properties that $z_i(\mathbf{x}) > z_i(\mathbf{y}) \Rightarrow P_i(\mathbf{x}, \mathbf{y}) < 1$ and that A is a strictly monotone function are quite natural properties since by this we simply require that the order P should react to any change of any of the functions z_i. Thus for practical applications we suggest to use fuzzy orders and aggregation functions respecting these properties.

6 Numerical Example

Let us observe the following linear programming problem:

$$\max\ z_1 = x_1, \quad \max\ z_2 = x_2,$$
$$\text{s.t. } x_1 + x_2 \leq 1, \quad x_1, x_2 \geq 0.$$

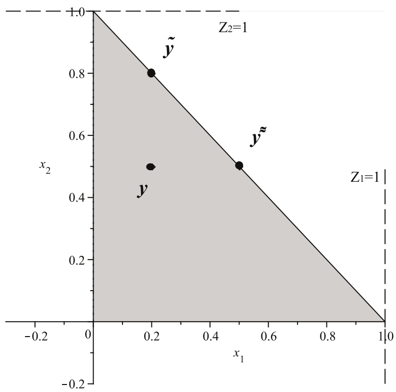

Figure 1. The solution space.

Figure 1 shows the solution space of this problem, where we coloured in gray the feasible region of the problem and dotted lines denote the level lines of the objective functions for which the corresponding objective reaches its maximum. We have chosen the simple (in the sense of input data) problem in order not to pay attention into details of computation but to illustrate the naturality of the proposed approach. Here we demonstrate the computation and how the result depends on the choice of an aggregation function and the base t-norm.

The point $(1, 0)$ is optimal solution with respect to the objective function z_1, the point $(0, 1)$ is the optimal solution with respect to the objective function z_2. Obviously the set $\{(x_1, x_2)\colon x_1 \in [0, 1],\ x_2 = 1 - x_1\}$ is the set of Pareto optimal solutions.

We follow the approach described above and apply the following fuzzy order relations based on Łukasiewicz t-norm (see Example 3):

$$P_1(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & \text{if } x_1 \leq y_1, \\ 1 - x_1 + y_1, & \text{otherwise,} \end{cases}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$;

$$P_2(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & \text{if } x_2 \leq y_2, \\ 1 - x_2 + y_2, & \text{otherwise.} \end{cases}$$

Further we aggregate the corresponding fuzzy order relations with the aid of the following aggregation function: $A(x, y) = (x + y)/2$, which is an aggregation function preserving T_L-transitivity. Thus:

$$P(\mathbf{x},\mathbf{y}) = A\big(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y})\big) = \frac{P_1(\mathbf{x},\mathbf{y}) + P_2(\mathbf{x},\mathbf{y})}{2}$$

Further we should solve the following problem:

$$\max_{\mathbf{y}\in D} \min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}).$$

Let us contract the set D for the simplicity of calculations, so we have to find a set B such that $B \subset D$ and

$$\max_{\mathbf{y}\in B} \min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in D} \min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}). \tag{6.1}$$

Let us prove that if $B = \{(x_1, x_2)\colon x_1 \in [0,1],\ x_2 = 1-x_1\}$ then Equation (6.1) holds. We start with the proof of the following equation:

$$\max_{\mathbf{y}\in B} \min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in D} \min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}).$$

Let us prove by contradiction, that is we suppose that $\mathbf{y} = (y_1, y_2) \in D$ but $\mathbf{y} \notin B$. Then there exist points $\tilde{\mathbf{y}} = (y_1, 1-y_1)$ and $\tilde{\tilde{\mathbf{y}}} = (1-y_2, y_2)$ (see Figure 1) such that

$$\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}) < \min_{\mathbf{x}\in D} P(\mathbf{x},\tilde{\mathbf{y}}) \quad \text{and}$$
$$\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}) < \min_{\mathbf{x}\in D} P(\mathbf{x},\tilde{\tilde{\mathbf{y}}}).$$

Let us prove the first equation. We will prove that for all $\mathbf{x} \in D$ it holds

$$A\big(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y})\big) \leq A\big(P_1(\mathbf{x},\tilde{\mathbf{y}}), P_2(\mathbf{x},\tilde{\mathbf{y}})\big)$$

and there exists $\bar{\mathbf{x}} \in D$ such that $A(P_1(\bar{\mathbf{x}},\mathbf{y}), P_2(\bar{\mathbf{x}},\mathbf{y})) < A(P_1(\bar{\mathbf{x}},\tilde{\mathbf{y}}), P_2(\bar{\mathbf{x}},\tilde{\mathbf{y}}))$.

We know that $P_1(\mathbf{x},\mathbf{y}) = P_1(\mathbf{x},\tilde{\mathbf{y}})$ since $z_1(\mathbf{y}) = z_1(\tilde{\mathbf{y}})$. Let us now compare $P_2(\mathbf{x},\mathbf{y})$ and $P_2(\mathbf{x},\tilde{\mathbf{y}})$. Obviously $z_2(\mathbf{y}) < z_2(\tilde{\mathbf{y}})$. Further we distinguish between the following three cases:

1. If $z_2(\mathbf{y}) < z_2(\tilde{\mathbf{y}}) \leq z_2(\mathbf{x})$ then $P_2(\mathbf{x},\mathbf{y}) = 1-|z_2(\mathbf{x})-z_2(\mathbf{y})| < 1-|z_2(\mathbf{x})-z_2(\tilde{\mathbf{y}})| = P_2(\mathbf{x},\tilde{\mathbf{y}})$.

2. If $z_2(\mathbf{x}) \leq z_2(\mathbf{y}) < z_2(\tilde{\mathbf{y}})$ then $P_2(\mathbf{x},\mathbf{y}) = P_2(\mathbf{x},\tilde{\mathbf{y}}) = 1$ since $z_2(\mathbf{x}) \leq z_2(\mathbf{y})$ and $z_2(\mathbf{x}) < z_2(\tilde{\mathbf{y}})$.

3. If $z_2(\mathbf{y}) < z_2(\mathbf{x}) < z_2(\tilde{\mathbf{y}})$ then $P_2(\mathbf{x},\mathbf{y}) \leq P_2(\mathbf{x},\tilde{\mathbf{y}})$ since $P_2(\mathbf{x},\tilde{\mathbf{y}}) = 1$.

Thus for all $\mathbf{x} \in D$ $A(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y})) \leq A(P_1(\mathbf{x},\tilde{\mathbf{y}}), P_2(\mathbf{x},\tilde{\mathbf{y}}))$, although there obviously exists $\bar{\mathbf{x}} \in D$ such that $z_2(\mathbf{y}) < z_2(\bar{\mathbf{x}}) < z_2(\tilde{\mathbf{y}})$. In this case $P_2(\bar{\mathbf{x}},\tilde{\mathbf{y}}) = 1$ but $P_2(\bar{\mathbf{x}},\mathbf{y}) < 1$. Thus, because of the strict monotonicity of the function A

$$A\big(P_1(\bar{\mathbf{x}},\mathbf{y}), P_2(\bar{\mathbf{x}},\mathbf{y})\big) < A\big(P_1(\bar{\mathbf{x}},\tilde{\mathbf{y}}), P_2(\bar{\mathbf{x}},\tilde{\mathbf{y}})\big).$$

By this we have finished the proof that

$$\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in D}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}).$$

We continue with the proof of the following equation:

$$\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in B}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y}).$$

Let us prove by contradiction, that is we suppose that $\mathbf{x} = (x_1, x_2) \in D$ but $\mathbf{x} \notin B$. Then there exist points $\tilde{\mathbf{x}} = (x_1, 1 - x_1)$ and $\tilde{\tilde{\mathbf{x}}} = (1 - x_2, x_2)$ such that for the fixed $\mathbf{y} \in B$

$$P(\tilde{\mathbf{x}},\mathbf{y}) < P(\mathbf{x},\mathbf{y}) \quad \text{and} \quad P(\tilde{\tilde{\mathbf{x}}},\mathbf{y}) < P(\mathbf{x},\mathbf{y}).$$

Let us prove the first equation. We will prove that for all $\mathbf{y} \in B$

$$A(P_1(\tilde{\mathbf{x}},\mathbf{y}), P_2(\tilde{\mathbf{x}},\mathbf{y})) \le A(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y})).$$

Obviously $P_1(\tilde{\mathbf{x}},\mathbf{y}) = P_1(\mathbf{x},\mathbf{y})$ since $z_1(\tilde{\mathbf{x}}) = z_1(\mathbf{x})$.

Let us now compare $P_2(\mathbf{x},\mathbf{y})$ and $P_2(\tilde{\mathbf{x}},\mathbf{y})$. Obviously $z_2(\mathbf{x}) < z_2(\tilde{\mathbf{x}})$. Further we distinguish between the following three cases:

1. If $z_2(\mathbf{x}) < z_2(\tilde{\mathbf{x}}) \le z_2(\mathbf{y})$ then $P_2(\mathbf{x},\mathbf{y}) = P_2(\tilde{\mathbf{x}},\mathbf{y}) = 1$ since $z_2(\mathbf{x}) < z_2(\mathbf{y})$ and $z_2(\tilde{\mathbf{x}}) \le z_2(\mathbf{y})$.
2. If $z_2(\mathbf{y}) \le z_2(\mathbf{x}) < z_2(\tilde{\mathbf{x}})$ then $P_2(\tilde{\mathbf{x}},\mathbf{y}) = 1 - |z_2(\tilde{\mathbf{x}}) - z_2(\mathbf{y})| < 1 - |z_2(\mathbf{x}) - z_2(\mathbf{y})| = P_2(\mathbf{x},\mathbf{y})$.
3. If $z_2(\mathbf{x}) < z_2(\mathbf{y}) < z_2(\tilde{\mathbf{x}})$ then $P_2(\tilde{\mathbf{x}},\mathbf{y}) \le P_2(\mathbf{x},\mathbf{y})$ since $P_2(\mathbf{x},\mathbf{y}) = 1$.

Thus for all $\mathbf{y} \in B$ it holds $A(P_1(\tilde{\mathbf{x}},\mathbf{y}), P_2(\tilde{\mathbf{x}},\mathbf{y})) \le A(P_1(\mathbf{x},\mathbf{y}), P_2(\mathbf{x},\mathbf{y}))$, although there obviously exists $\bar{\mathbf{y}}$ such that $z_1(\mathbf{x}) < z_1(\bar{\mathbf{y}}) < z_1(\tilde{\mathbf{x}})$. In this case $P_1(\mathbf{x},\bar{\mathbf{y}}) = 1$ but $P_1(\tilde{\mathbf{x}},\bar{\mathbf{y}}) < 1$. Thus, because of the strict monotonicity of the function A

$$A\big(P_1(\tilde{\mathbf{x}},\bar{\mathbf{y}}), P_2(\tilde{\mathbf{x}},\bar{\mathbf{y}})\big) < A\big(P_1(\mathbf{x},\bar{\mathbf{y}}), P_2(\mathbf{x},\bar{\mathbf{y}})\big).$$

Thus $\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in D}\min_{\mathbf{x}\in D} P(\mathbf{x},\mathbf{y})$, where B is the set of Pareto optimal solutions. It is an important fact which makes calculations much easier.

Let us come back to our initial example and by the following figures we demonstrate the dependences of the value $\min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y})$ on the choice of $\mathbf{y}$. The horizontal axises are the set B of Pareto optimal solutions: $B = \{(y_1, y_2)\colon y_1 \in [0,1],\ y_2 = 1 - y\}$, where the elements $\mathbf{y} = (y_1, y_2)$ of the set B are presented by its first coordinate:

Figure 2 and Figure 3 demonstrate the results when we use Łukasiewicz t-norm and $A(a_1, a_2) = (a_1 + a_2)/2$ and $A(a_1, a_2) = (a_1 + 2a_2)/3$ respectively. The results are rather expected: when the weights are the same (1/2 and 1/2) the maximum point is exactly in the middle, but if the weights are 1/3 and 2/3 then the maximum point divides the unit interval respectively as 1/3 and 2/3. The results for the problem $\max_{\mathbf{y}\in B}\min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y})$ are the same when we use the product t-norm, but the shape of the function $f(\mathbf{y}) = \min_{\mathbf{x}\in B} P(\mathbf{x},\mathbf{y})$ is slightly different, see Figure 4 and Figure 5.

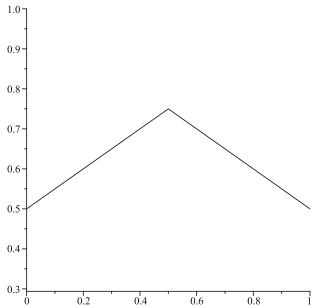

Figure 2. $f(\mathbf{y})$, where $A(a_1, a_2) = \frac{a_1+a_2}{2}$

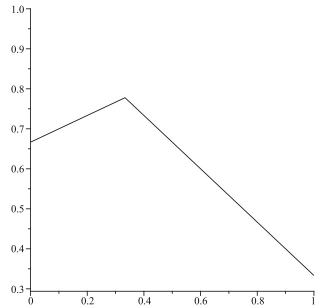

Figure 3. $f(\mathbf{y})$, where $A(a_1, a_2) = \frac{a_1+2a_2}{3}$

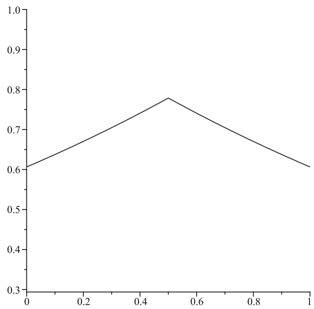

Figure 4. $f(\mathbf{y})$, where $A(a_1, a_2) = a_1^{\frac{1}{2}} a_2^{\frac{1}{2}}$

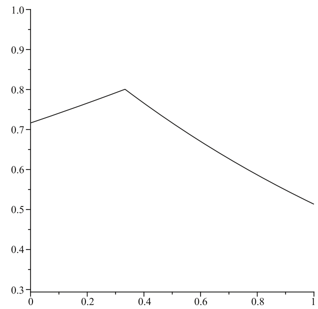

Figure 5. $f(\mathbf{y})$, where $A(a_1, a_2) = a_1^{\frac{1}{3}} a_2^{\frac{2}{3}}$

7 Conclusions

In our paper we proposed a solution approach for multi-objective linear programming problem where we have used fuzzy order relations instead of the membership functions prescribing the satisfaction degree of reaching the solution of individual problems. Further, to get an optimal compromise solution the fuzzy order relations were aggregated and the "maximum" with respect to the aggregated fuzzy order relation has been found. Although the approach described in our paper is more complicated in computations it has the following advantages:

1. This approach generalizes the classical linear programming approach and testifies its naturality.

2. There is a reasonable explanation of the choice of the "shape" of fuzzy order relation. In classical fuzzy approach more often the choice of linear membership functions is not explained or is explained by "facilitation computation for obtaining solutions". The choice in our approach is caused by the fuzzy environment (or t-norm) in which we are working.

3. There is a reasonable explanation of the choice of aggregation function. The choice in our approach is caused by the necessity to preserve the properties of initial fuzzy order relations.

We see the following two possible directions for future research:

1. We see that in our example presented in Section 6 the results do not depend on the choice of a t-norm. It is interesting to investigate how the choice of a t-norm affects the results in general.

2. The usage of fuzzy order relations are investigated only for the simplest fuzzy approach for solving multi-objective linear programming problems. It is interesting also to involve fuzzy order relations for two-level (multi-level) linear programming problems. Also it is interesting to realize interactive fuzzy programming or fuzzy compromise approach by involving fuzzy order relations.

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