

A Splitting Preconditioner for the Incompressible Navier–Stokes Equations*

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Abstract. In this paper, a splitting preconditioner based on the relaxed dimensional factorization (RDF) preconditioner and the modified augmented Lagrangian (MAL) preconditioner for the incompressible Navier–Stokes equations is presented. The preconditioned matrix is analyzed, and similar results arising from the RDF and the MAL preconditioners are obtained. The corresponding details of the spectrum analysis are given. Finally, we compare the three preconditioners and numerical experiments are implemented by using the IFISS package.

Keywords: saddle point problem, Krylov subspace method, splitting preconditioner.

AMS Subject Classification: 65N22; 65F08.

1 Introduction

We consider the following incompressible Navier–Stokes equations describing the flow of viscous Newtonian fluids:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{on } \Omega \times (0, T], \quad (1.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{on } \Omega \times [0, T], \quad (1.2)$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \times [0, T], \quad (1.3)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{on } \Omega, \quad (1.4)$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is an open bounded domain with boundary $\partial\Omega$, $[0, T]$ is the time interval, the unknown velocity fields $\mathbf{u}(\mathbf{x}, t)$ and pressure fields

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$p(\mathbf{x}, t)$, ν is the kinematic viscosity, Δ is the vector Laplacian operator, ∇ is the gradient operator, div is the divergence, \mathbf{f} , \mathbf{g} and \mathbf{u}_0 are given functions. After implicit time discretization and linearization of the Navier–Stokes system of equations by Picard fixed-point iteration, we get a sequence of the Oseen problems. Discretization of the Oseen problems using finite element methods results in a sequence of large sparse linear systems of equations. These equations are expressed as

$$\mathbf{H}\mathbf{x} = b \tag{1.5}$$

with

$$\mathbf{H} = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} f \\ g \end{pmatrix},$$

with \mathbf{u} and p representing the discrete velocity and pressure, respectively. A denotes the discretization of the diffusion, convection and time-dependent terms. A is a diagonal block matrix, e.g. $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ in 2D. B^T is the discrete gradient, B denotes the (negative) discrete divergence, C is a stabilization matrix and depends on the discretization stability condition, f and g contain the forcing and boundary terms. If we use the LBB-stable finite element to discretize this problem and use a simple transformation $J = \begin{pmatrix} I_A & 0 \\ 0 & -I_C \end{pmatrix}$, where I_A and I_C are the identity matrices, then (1.5) can be rewritten in the mathematically equivalent system as

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{1.6}$$

where the spectrum of the coefficient matrix of (1.6) is entirely contained in the right half plane (see [3]).

These systems can be solved by direct methods, but they require extensive resources in terms of computational time and memory. For 3D and large 2D problems, iterative methods, in combination with suitable preconditioners, are the methods of choice. Benzi *et. al* have presented the RDF preconditioner to effectively solve the system (1.6) in [5]. The RDF preconditioner is one kind of dimensional splitting (DS) preconditioners [4, 8, 10] and is more effective than the preconditioner derived in [4]. The experimentally optimal parameter within the RDF preconditioner always strongly depends on mesh size. Although the parameter values can be estimated by Fourier Analysis, see [5], they become smaller and smaller for finer grids and cannot perfectly approximate the experimentally optimal values. However, according to the structure of the RDF preconditioner, practical implementation of the RDF preconditioner is good. The augmented Lagrangian (AL) preconditioner has been presented by Benzi *et. al* and it is based on the block triangular preconditioner for the augmented system of (1.6), see [3, 6, 7, 9, 14, 15].

For the AL preconditioner, approximating the Schur complement $BA^{-1}B^T$ is relatively easy, and the main issue is how to effectively compute solution of linear systems associated with the augmented block; see Section 4. The modified AL preconditioner is presented to decrease computational cost of the AL preconditioner, but it may be an inferior position comparing with the RDF preconditioner because more matrix-vector multiplications within the augmented

system make costs increase. However, the MAL preconditioner retains a portion of features of the AL preconditioner. The parameter of the MAL preconditioner is almost steady and robust, it is largely insensitive to problem parameters including grid size, viscosity ν , non-uniform meshes, etc. The confidence interval of this parameter is wide, the optimal parameter value obtained by Fourier analysis [5,9] can be close to the experimentally optimal value of the MAL preconditioner. Our starting point is to investigate the sensitivity of the parameter of these preconditioners, and to search a preconditioner satisfying these advantages. That is, we consider whether there is a preconditioner which can be combined with both RDF and MAL preconditioners. In this paper we propose a splitting preconditioner, whose ideas come from the RDF preconditioner and the MAL preconditioner. Moreover, we analyze the spectrum distribution of the corresponding preconditioned matrix and the choice of the optimal parameter.

The rest of the paper is organized as follows. In Sections 2 and 3, we briefly introduce the RDF preconditioner and the MAL preconditioner respectively. In Section 4, we present our splitting preconditioner and derive some properties. Then we show how Fourier analysis can be used to select the parameter. In Section 5, we present results of numerical experiments including comparisons with the RDF and the MAL preconditioners. Finally, some conclusions are drawn in Section 6.

2 The RDF Preconditioner

In this section, we make a brief description of the RDF preconditioner; for further details, see [4, 5]. For simplicity, we only consider the 2D case. The reformulation of (1.6) is expressed as

$$H = \begin{pmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & 0 \end{pmatrix}, \quad (2.1)$$

where $H \in \mathbb{R}^{N \times N}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{m \times n_i}$ and $\sum_{i=1}^2 n_i + m = N$. The RDF preconditioner in [5] is defined as follows

$$P_{RDF} = \begin{pmatrix} A_1 & -\frac{1}{\tau} B_1^T B_2 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & \tau I \end{pmatrix}, \quad (2.2)$$

where τ is the positive parameter. The RDF preconditioner is also scaled, a scaling is applied to the coefficient matrix before forming the preconditioner. The behaviour of RDF preconditioning can be improved by diagonal scaling. Unless otherwise specified, we always perform a preliminary symmetric scaling of the system $Hx = b$ in the form $D^{-\frac{1}{2}} H D^{-\frac{1}{2}} y = D^{-\frac{1}{2}} b$, with $y = D^{\frac{1}{2}} x$ and $D = \text{diag}(D_1, D_2, I_m)$, where $\text{diag}(D_1, D_2)$ is the main diagonal of the velocity mass matrix for 2D problems. Obviously, this diagonal scaling is regarded as a simple preconditioner.

3 The Modified Augmented Lagrangian Preconditioner

For the steady-state problems (1.6), the equivalent AL formulation [14, 15] is given by

$$H_\gamma \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} A_\gamma & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f_\gamma \\ -g \end{pmatrix}, \tag{3.1}$$

where $A_\gamma = A + \gamma B^T W^{-1} B$, $f_\gamma = f + \gamma B^T W^{-1} g$, W is SPD and $\gamma > 0$. Thus the ideal AL system and the ideal AL preconditioner [6, 7, 9] are

$$\begin{pmatrix} A_\gamma & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f_\gamma \\ g \end{pmatrix}, \quad P_{al} = \begin{pmatrix} A_\gamma & B^T \\ 0 & \hat{S}_{al} \end{pmatrix},$$

where \hat{S}_{al} is the approximate Schur complement.

In order to retain the transformation property of the system (1.6), in this paper, the proposed P_{al} -type AL preconditioner is the block triangular matrix as follows

$$P_{AL} = \begin{pmatrix} A_\gamma & 0 \\ -B & \hat{S} \end{pmatrix}, \tag{3.2}$$

where \hat{S} is the approximate Schur complement and usually implicitly defined by

$$\hat{S}^{-1} = \nu M_p^{-1} + \gamma W^{-1},$$

where M_p denotes the approximate pressure mass matrix, ν is the viscosity. A good choice of W is the pressure mass matrix \hat{M}_p , see [6, 14, 16]. In practice, we always use the main diagonal of the pressure mass matrix. In many cases, W is also replaced by M_p . For decreasing calculation cost, in practice, M_p is a diagonal matrix or is replaced by spectrally equivalent diagonal matrix.

Considering 2D case, we have $A = \text{diag}(A_1, A_2)$ and $B = (B_1, B_2)$, where $H_\gamma \in \mathbb{R}^{N \times N}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{m \times n_i}$ and $\sum_{i=1}^2 n_i + m = N$. $A + \gamma B^T W^{-1} B$ is partitioned into 2×2 square block matrix, i.e.,

$$\begin{aligned} A_\gamma &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} B_1^T \\ B_2^T \end{pmatrix} W^{-1} \begin{pmatrix} B_1 & B_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 + \gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & A_2 + \gamma B_2^T W^{-1} B_2 \end{pmatrix} \\ &\approx \begin{pmatrix} A_1 + \gamma B_1^T W^{-1} B_1 & 0 \\ \gamma B_2^T W^{-1} B_1 & A_2 + \gamma B_2^T W^{-1} B_2 \end{pmatrix}. \end{aligned}$$

Then the (1, 2) block is dropped, the modified augmented Lagrangian preconditioner is obtained. Readers can refer to [6, 7, 9] for further details including the choice of matrix W . Throughout this paper, we always consider the MAL preconditioner.

4 A Splitting Preconditioner

In this section, we define a preconditioner via the splitting scheme. Our splitting preconditioner is formulated as follows

$$\begin{aligned} H &= \begin{pmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & 0 \end{pmatrix} = P - R \\ &= \begin{pmatrix} A_1 & -\alpha B_1^T W_x^{-1} B_2 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & \alpha^{-1} W_x \end{pmatrix} - \begin{pmatrix} 0 & -\alpha B_1^T W_x^{-1} B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha^{-1} W_x \end{pmatrix}, \end{aligned}$$

where W_x is SPD and $\alpha > 0$,

$$P = \begin{pmatrix} A_1 & -\alpha B_1^T W_x^{-1} B_2 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & \alpha^{-1} W_x \end{pmatrix}. \quad (4.1)$$

By comparing (4.1) with (2.2) and (3.2), it is shown that the block structure of these preconditioners is similar. We can find that W_x and W_x^{-1} are added into the RDF preconditioner on the one hand, then can get (4.1); on the other hand, dealing with the MAL preconditioner also yields (4.1). Obviously, (4.1) is not a type of the dimensional factorization preconditioner, it is just a generic splitting type. Based on this relationship, we consider $W_x = W$, and don't pursue other choices of W_x .

Now we analyze some properties of new preconditioner. Firstly, we consider the spectral properties of the new preconditioned matrix HP^{-1} , then we resort into Fourier analysis for guiding in the choice of the parameter α . More details which are similar to properties of the RDF preconditioner and the MAL preconditioner are given; see [5, 7, 9].

Lemma 1. *Assume $\hat{A}_1 = A_1 + \alpha B_1^T W^{-1} B_1$, $\hat{A}_2 = A_2 + \alpha B_2^T W^{-1} B_2$ are invertible. Obviously,*

$$\begin{aligned} P &= \begin{pmatrix} A_1 & 0 & \alpha B_1^T W^{-1} \\ 0 & I & 0 \\ -B_1 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & A_2 & B_2^T \\ 0 & -B_2 & \alpha^{-1} W \end{pmatrix} = \begin{pmatrix} I & 0 & \alpha B_1^T W^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ &\times \begin{pmatrix} \hat{A}_1 & 0 & 0 \\ 0 & I & 0 \\ -B_1 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & \hat{A}_2 & B_2^T \\ 0 & 0 & \alpha^{-1} W \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\alpha W^{-1} B_2 & I \end{pmatrix}, \end{aligned}$$

and we note $S_1 = B_1 \hat{A}_1^{-1} B_1^T$, $S_2 = B_2 \hat{A}_2^{-1} B_2^T$, then

$$P^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \alpha W^{-1} B_2 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & \hat{A}_2^{-1} & -\alpha \hat{A}_2^{-1} B_2^T W^{-1} \\ 0 & 0 & \alpha W^{-1} \end{pmatrix}$$

$$\begin{aligned} & \times \begin{pmatrix} \hat{A}_1^{-1} & 0 & 0 \\ 0 & I & 0 \\ B_1 \hat{A}_1^{-1} & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & -\alpha B_1^T W^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ & = \begin{pmatrix} I & 0 & 0 \\ 0 & \hat{A}_2^{-1} & -\alpha \hat{A}_2^{-1} B_2^T W^{-1} \\ 0 & \alpha W^{-1} B_2 \hat{A}_2^{-1} & \alpha W^{-1} - \alpha^2 W^{-1} S_2 W^{-1} \end{pmatrix} \\ & \times \begin{pmatrix} \hat{A}_1^{-1} & 0 & -\alpha \hat{A}_1^{-1} B_1^T W^{-1} \\ 0 & I & 0 \\ B_1 \hat{A}_1^{-1} & 0 & I - \alpha S_1 W^{-1} \end{pmatrix}. \end{aligned}$$

According to the special structure of (4.1), the new preconditioner can be factorized into some factors. It is convenient to implement numerical experiments and analyze its properties. In [5,17], there are some well known results. We extend those propositions for the preconditioned matrix of (4.1) as follows.

Lemma 2. *Assume*

$$V_{22} = \begin{pmatrix} \alpha^2 \hat{A}_2^{-1} B_2^T W^{-1} S_1 W^{-1} B_2 & -\hat{A}_2^{-1} B_2^T W^{-1} (W - \alpha S_1) \\ -\alpha^2 W^{-1} (I - \alpha S_2 W^{-1}) S_1 W^{-1} B_2 & W^{-1} (I - \alpha S_2 W^{-1}) (W - \alpha S_1) \end{pmatrix}.$$

Then V_{22} has zero eigenvalues with multiplicity at least n_2 , and the remaining eigenvalues are $1 - \lambda_i$, where λ_i are the eigenvalues of $m \times m$ matrix $Z_\alpha = \alpha(S_1 W^{-1} + S_2 W^{-1}) - 2\alpha^2 S_1 W^{-1} S_2 W^{-1}$.

Proof. Referring to [5], this proposition can be easily shown . \square

Theorem 1. *The preconditioned matrix HP^{-1} has eigenvalues equal to 1 with multiplicity at least $n_1 + n_2$, the remaining eigenvalues are the eigenvalues λ_i of the matrix Z_α .*

Proof. Referring to [5], this theorem can be easily proved. \square

We note that $Z_\alpha = \alpha(\hat{S}_1 + \hat{S}_2) - 2\alpha^2 \hat{S}_1 \hat{S}_2$, where $\hat{S}_1 = B_1 \hat{A}_1^{-1} B_1^T W^{-1}$, $\hat{S}_2 = B_2 \hat{A}_2^{-1} B_2^T W^{-1}$.

Theorem 2. *The eigenvalues of Z_α are of the form $\lambda_i = \frac{\mu_i}{\alpha + \mu_i}$, where the eigenvalues μ_i satisfy the generalized eigenvalue problem*

$$BA^{-1} B^T W^{-1} \varphi_i = \mu_i \left(\frac{1}{\alpha^2} I + \tilde{S}_1 \tilde{S}_2 \right) \varphi_i \quad \text{with } \tilde{S}_k = B_k A_k^{-1} B_k^T W^{-1}.$$

Proof. Applying the Sherman–Morrison–Woodbury formulation to \hat{S}_k gives $\hat{S}_k = \frac{1}{\alpha} \tilde{S}_k (\frac{1}{\alpha} I + \tilde{S}_k)^{-1}$, where $\tilde{S}_k = B_k A_k^{-1} B_k^T W^{-1}$. Then we get similar results to [5]. \square

These propositions show that there exist inseparable relationships between the RDF preconditioner and our splitting preconditioner. Assume all the idealized conditions are satisfied; see the parameter analysis part of this section. The

matrices A_1, A_2, B_1, B_2 and W are all diagonalizable by the discrete Fourier transform, i.e., $UA_iU^H = \text{diag}(a_i), UB_iU^H = \text{diag}(b_i), UWU^H = \text{diag}(w)$ with $i = 1, 2$. Here a_i, b_i, w are vectors containing the eigenvalues of the corresponding matrices, and the unitary matrix U is composed of Fourier components $\frac{e^{i2\pi h\theta k}}{\sqrt{l}}$ with $i = \sqrt{-1}, k = 1, 2, \dots, l, \theta = 1, 2, \dots, l$ and grid size $h = 1/l$, see [5]. Refer to the RDF preconditioner, let the matrices \hat{S}_1, \hat{S}_2 and Z_α can be diagonalized by U , then the eigenvalues of Z_α can be simply expressed as

$$z_\alpha = \alpha(s_1 + s_2) - 2\alpha^2s_1s_2,$$

where s_1, s_2 are the eigenvalues of \hat{S}_1, \hat{S}_2 and are respectively given by $s_1 = \frac{b_1^2}{a_1w + \alpha b_1^2}$ and $s_2 = \frac{b_2^2}{a_2w + \alpha b_2^2}$. Readers can refer to [5] for further details. According to [5], we can compare the corresponding eigenvalues formulation with respect to the RDF preconditioner, i.e., $z_\tau = \frac{1}{\tau}(s_1 + s_2) - \frac{2}{\tau^2}s_1s_2$, where $s_1 = \frac{b_1^2}{a_1 + \frac{1}{\tau}b_1^2}$, and $s_2 = \frac{b_2^2}{a_2 + \frac{1}{\tau}b_2^2}$. Let $\alpha \leq 1, \tau \leq 1$ and $w \leq 1$, it is not hard to see that the magnitude of $\frac{1}{\tau}$ and $\frac{1}{\tau^2}$ increase more quickly than α and α^2 . Then in conjunction with the above equation, we also can find the corresponding optimal value of (α, w) and τ so that the eigenvalues z_α have a more clustering effect than the eigenvalues z_τ , the worst case is that the clustering effect of the both eigenvalues is the same.

We also propose another method to analyze the spectrum of the preconditioned matrix in order to observe the relationship between our preconditioner and the MAL preconditioner. Z_α is expressed as

$$\begin{aligned} Z_\alpha &= \alpha(\hat{S}_1 + \hat{S}_2) - 2\alpha^2\hat{S}_1\hat{S}_2 \\ &= \alpha(B_1\hat{A}_1^{-1}B_1^T + B_2\hat{A}_2^{-1}B_2^T - 2\alpha B_1\hat{A}_1^{-1}B_1^TW^{-1}B_2\hat{A}_2^{-1}B_2^T)W^{-1} \\ &= \alpha(B_1 \ B_2) \begin{pmatrix} \hat{A}_1^{-1} & -2\alpha\hat{A}_1^{-1}B_1^TW^{-1}B_2\hat{A}_2^{-1} \\ 0 & \hat{A}_2^{-1} \end{pmatrix} \begin{pmatrix} B_1^T \\ B_2^T \end{pmatrix} W^{-1}. \end{aligned}$$

Note $\hat{A}_\alpha = A_\alpha + \alpha B^TW^{-1}B$, where $A_\alpha = \begin{pmatrix} A_1 & \alpha B_1^TW^{-1}B_2 \\ -\alpha B_2^TW^{-1}B_1 & A_2 \end{pmatrix}$, then $Z_\alpha = \alpha B\hat{A}_\alpha^{-1}B^TW^{-1}$.

Lemma 3. Let $A_\alpha = \begin{pmatrix} A_1 & A_{12} \\ -A_{12}^T & A_2 \end{pmatrix} \in \mathbb{R}^{n \times n}, n = n_1 + n_2, A_1 \in \mathbb{R}^{n_1 \times n_1}, A_2 \in \mathbb{R}^{n_2 \times n_2}$ and A_1, A_2 are positive definite. Then A_α is positive definite.

Lemma 4. Let $A_\alpha \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}(m < n)$ and $\alpha \in \mathbb{R}$. Assume that matrices $A_\alpha, A_\alpha + \alpha B^TW^{-1}B, BA_\alpha^{-1}B^T$ and $B(A_\alpha + \alpha B^TW^{-1}B)^{-1}B^T$ are all invertible. Then $(B(A_\alpha + \alpha B^TW^{-1}B)^{-1}B^T)^{-1} = (BA_\alpha^{-1}B^T)^{-1} + \alpha W^{-1}$.

Since proofs of Lemmas 3 and 4 are easy, readers can refer to [6]. The conditions of Lemma 4 are satisfied if we assume that B has full row rank and A_α is positive definite. Hence the remaining m eigenvalues λ_i are solutions of the generalized eigenproblem $\alpha B\hat{A}_\alpha^{-1}B^T\phi_i = \lambda_i W\phi_i$. Lemma 4 yields

$$\begin{aligned} \alpha\lambda_i^{-1}\phi_i &= (B\hat{A}_\alpha^{-1}B^T)^{-1}W\phi_i = (B(A_\alpha + \alpha B^TW^{-1}B)^{-1}B^T)^{-1}W\phi_i \\ &= ((BA_\alpha B^T)^{-1}W + \alpha I)\phi_i = (\hat{\mu}_i^{-1} + \alpha)\phi_i, \end{aligned}$$

$\alpha\lambda_i^{-1} = \hat{\mu}_i^{-1} + \alpha$, where $\hat{\mu}_i$ satisfies the generalized eigenproblem $BA_\alpha^{-1}B^T\phi_i = \hat{\mu}_iW\phi_i$. Hence, the non-unit eigenvalues of the preconditioned matrix are given as follows:

Theorem 3. *The eigenvalues λ_i of Z_α are of the form $\lambda_i = \frac{\alpha\hat{\mu}_i}{1+\alpha\hat{\mu}_i}$, where the $\hat{\mu}_i$'s satisfy the generalized eigenvalue problem $BA_\alpha^{-1}B^T\phi_i = \hat{\mu}_iW\phi_i$.*

Now we analyze boundary of λ_i , some corresponding results and relations are obtained. Writing $\lambda_i = a_\lambda + ib_\lambda$ and $\hat{\mu}_i = a_\mu + ib_\mu$, it is easy to obtain the following expressions of the real and the imaginary parts of λ_i :

$$\Re(\lambda_i) = \frac{\alpha(a_\mu + \alpha(a_\mu^2 + b_\mu^2))}{(\alpha a_\mu + 1)^2 + (\alpha b_\mu)^2}, \quad \Im(\lambda_i) = \frac{\alpha b_\mu}{(\alpha a_\mu + 1)^2 + (\alpha b_\mu)^2}. \quad (4.2)$$

Theorem 4. *The remaining m eigenvalues λ_i are given by Theorem 2, where $\hat{\mu}_i = a_\mu + ib_\mu$ satisfies (4.2). The following estimates in [2, 6, 13]:*

$$0 < \min_i \frac{\alpha a_\mu}{1 + \alpha a_\mu} \leq a_\lambda \leq 1, \quad |b_\lambda| \leq \max_\mu \min\left(\alpha|b_\mu|, \frac{1}{\alpha|b_\mu|}\right) \leq 1$$

are also valid for the eigenvalues of the matrix Z_α .

It is shown that our preconditioner contains some features of the MAL preconditioner. We consider the matrix

$$A_\alpha = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} 0 & A_{12} \\ -A_{12}^T & 0 \end{pmatrix} = A + K$$

directly, it can get $\min \operatorname{Re}\{\lambda(A)\} \leq \operatorname{Re}\{\lambda(A_\alpha)\} \leq \max \operatorname{Re}\{\lambda(A)\}$ and the eigenvalues of K are purely imaginary type. Here we use the foregoing notation, i.e., $UA_\alpha U^H = \operatorname{diag}(d_\alpha)$, $UAU^H = \operatorname{diag}(d)$ and $UKU^H = \operatorname{diag}(ik)$, then the eigenvalues of $BA_\alpha^{-1}B^TW^{-1}$ and $BA^{-1}B^TW^{-1}$ are respectively given by $\hat{\mu} = \frac{b^2}{wd_\alpha}$ and $\mu = \frac{b^2}{wd}$. Therefore, if $0 \leq k^2 < d^2$, then $\operatorname{Re}\{\mu\} \leq \operatorname{Re}\{\hat{\mu}\} = \frac{b^2}{w(d - \frac{k^2}{d})}$, but the purely imaginary eigenvalues of K and the parameter factors may lead to the eigenvalues of A_α more decentralized, i.e., the eigenvalues of the corresponding preconditioned matrix have less clustering effect. According to analysis (Theorem 3 and [7]) and experiments, the eigenvalues with respect to our preconditioner tend to less clustering effect than the MAL preconditioner, this feature may cause the convergence rates of iterative method become slow.

In Figures 1 and 2, the conclusion of the foregoing analysis are confirmed. The SPP denotes our splitting preconditioner, and we show the spectrum plots of the preconditioned matrices obtained from the RDF, the SPP, the MAL and the AL preconditioners under the experimentally optimal values of τ , α , and γ respectively. We can see that the eigenvalues of the preconditioned matrix of the new preconditioner tend to more clustering effect compared with the RDF preconditioner, clustering effect of the eigenvalues of the preconditioned matrix of the AL preconditioner is the best. From plots, we can see that our preconditioner and the MAL preconditioner are slightly similar, this explains the fact that the convergence features of our preconditioner tend to the MAL

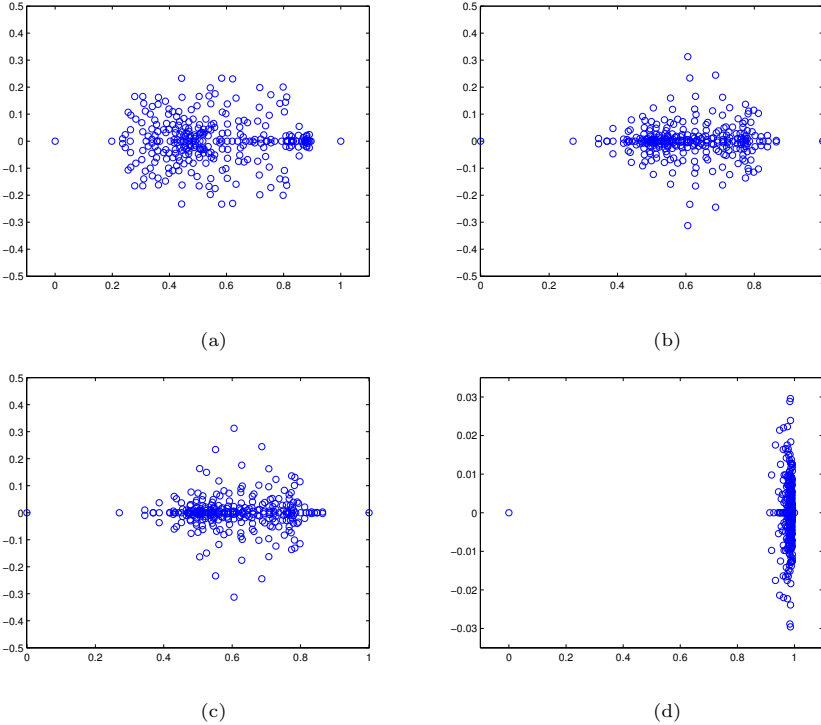


Figure 1. Spectrum of the preconditioned steady Oseen matrix, Q2-Q1 FEM, 32×32 uniform grid, $\nu = 0.01$, and experimentally optimal τ, α, γ . (a): the RDF, (b): the SPP, (c): the MAL, (d): the AL.

preconditioner. We note that the following numerical experiments verify this point. Thus our preconditioner is a compromise preconditioner.

Allowing $\alpha \rightarrow 0+$ or $\alpha \rightarrow \infty$ doesn't make the norm $\|H - P\|_2$ very small, then it suggests that there is an optimal α for our preconditioner. Hence, we resort to Fourier analysis (FA) for approximating the optimal value of the parameter α ; see [5, 9]. As usual, the use of this technique needs some rather drastic simplifications and assumptions about the problem. We assume that the Oseen problem has constant coefficients, it is defined on the unit square with periodic boundary conditions, and is discretized on a uniform, with grid size $h = 1/l$. Moreover, we assume the matrices A_1, A_2, B_1, B_2 and W are all square and commute. Though these assumptions are virtually never met in real problems, we emphasize that these assumptions are made to guide the choice of the parameter, and the parameter obtained by Fourier analysis often gives good results for more general problems [5, 9, 11].

To further simplify, we resort to the symbols of the corresponding operators. The block matrix A_i is a discrete scalar convection-diffusion-reaction operator

$$A_i = \sigma M + \nu L + N_i,$$

we mainly consider $\nu L + N_i$ operator, L is discrete Laplacians and N_i is discrete

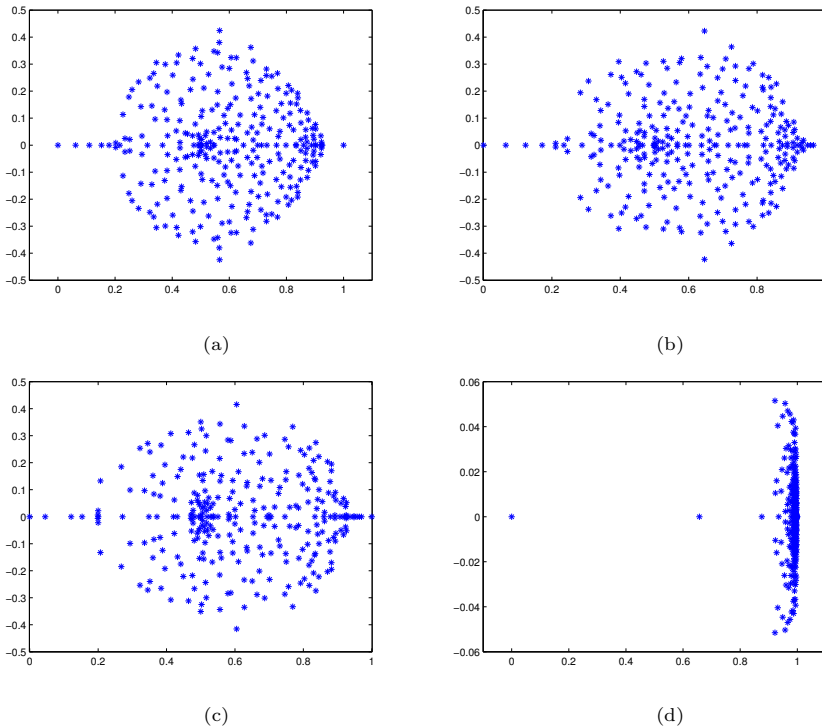


Figure 2. Spectrum of the preconditioned steady Oseen matrix, Q2-Q1 FEM, 32×32 uniform grid, $\nu = 0.001$, and experimentally optimal τ, α, γ . (a): the RDF, (b): the SPP, (c): the MAL, (d): the AL.

convection operators. Then the discrete 2D steady state convection-diffusion operator $\mathfrak{N} = I_l \otimes (\nu L_x + N_x) + (\nu L_y + N_y) \otimes I_l$, where \otimes denotes the tensor product, I_l is the identity matrix with l order, L_x and L_y are discrete 1D Laplacians, N_x and N_y are discrete 1D convection operators in the x and y directions. Similarly, the matrix B_i represent discrete partial derivatives with respect to x and y . Noting the discretization of the ordinary derivatives $\frac{d}{dx}$ and $\frac{d}{dy}$ by R_x and R_y , then $B_1 = I_l \otimes R_x$ and $B_2 = R_y \otimes I_l$. Let us assume that diffusion and convection terms are discretized by centered finite differences and the divergence is discretized by one-sided differences, and observe that W scales as h^2 , then we get the following correspondence between the operators and their symbols, $\theta = (\theta_x, \theta_y)$:

$$L = 2 - e^{i2\pi h\theta} - e^{-i2\pi h\theta}, \quad N = e^{i2\pi h\theta} - e^{-i2\pi h\theta},$$

$$R = h(1 - e^{i2\pi h\theta}), \quad W = h^2.$$

We note that the symbol respect the scaling of the matrices discretized by finite element methods, then L, N, R can be expressed as diagonal matrices, whose diagonal entries are the corresponding eigenvalues [9]. Hence matrices A_i and B_i can be represented by symbols as well. Moreover, we express $B_1 A_1^{-1} B_1^T W^{-1}, B_2 A_2^{-1} B_2^T W^{-1}$ as diagonal matrices $D_1 = \text{diag}(d_1)$,

$D_2 = \text{diag}(d_2)$, respectively. From [9], matrix $Z_\alpha = \alpha(S_1W^{-1} + S_2W^{-1}) - 2\alpha^2S_1W^{-1}S_2W^{-1}$ can be expressed as

$$Z_\alpha = (I + \alpha D_1)^{-1} + (I + \alpha D_2)^{-1} - 2(I + \alpha D_1)^{-1}(I + \alpha D_2)^{-1},$$

then the eigenvalues of Z_α are given by

$$\lambda(Z_\alpha) = \frac{\alpha d_1 + \alpha d_2}{1 + \alpha d_1 + \alpha d_2 + \alpha^2 d_1 d_2}.$$

Therefore, we want the eigenvalues of Z_α to be around 1, it is equivalent to solving the following optimization problem

$$\begin{aligned} & \min_{\alpha > 0} |\lambda(Z_\alpha) - 1|, \\ & \text{subject to } \theta = 1, 2, \dots, l. \end{aligned}$$

Comparing with the choice of the parameter in the MAL preconditioner [9], it is shown that these both minimum problems are equivalent. Thus we can turn to the choice of the parameter of the MAL preconditioner to obtain the parameter of our splitting preconditioner.

5 Numerical Experiments

In this section, we shall present numerical experiments for the linear systems coming from the two dimensional Oseen models of the incompressible flow discretized by finite element methods to test the performance of our preconditioner. Two main test problems are generated by IFISS software package [12]: Lid-driven cavity (LCD) problem and backward facing-step (BFS) problem. These experiments were performed in MATLAB 7 on an Intel (R) Pentium (R) 4 CPU with 3.00 GHz and 1GB of memory.

We should mention using restarted GMRES [18] as the Krylov subspace solver, stopped when the relative residual norm is reduced below 10^{-6} , the maximum subspace dimension is set to 20, in the tests it always uses zero initial guesses. Unless otherwise specified, the corresponding parameter of discretization is default in the IFISS software package. We also set $W = M_p = \text{diag}(\hat{M}_p)$, where \hat{M}_p denotes the pressure mass matrix in our numerical experiments, exact solves are performed on the subsystems involving \hat{A}_1 and \hat{A}_2 by means of direct *LU* factorization (for Oseen problems) after proceeding by AMD re-ordering technique [1, 5, 7, 9].

We consider the 2D leaky-lid driven cavity problem discretized by Q2-Q1 and Q2-P1 finite elements on the uniform and stretched grids. We are interested in observing the effect of the backward facing step problem on a non-square domain. In the given tables and figures, RDF denotes the RDF preconditioner, MAL is the modified AL preconditioner, SPP denotes our splitting preconditioner.

Example 1. The steady cavity Oseen problem. In this example, we perform the diagonal scaling technique for all preconditioning. In Tables 1–5, we present

Table 1. Number of GMRES(20) iterations and timings with the RDF, MAL and SPP preconditioners for steady Oseen problem (Q2-Q1 FEM uniform grids), the optimal parameters values of τ, α, γ are respectively found experimentally.

Grid	Viscosity								
	0.1			0.01			0.001		
	RDF	SPP	MAL	RDF	SPP	MAL	RDF	SPP	MAL
16×16	11	9	9	13	10	9	22	18	17
Setup time	0.0625	0.0781	0.0781	0.0625	0.0781	0.0781	0.0625	0.0781	0.0781
Iter time	0.0938	0.0781	0.2188	0.1094	0.0938	0.2344	0.1719	0.1250	0.3438
32×32	11	9	9	12	10	10	27	21	20
Setup time	0.3125	0.3281	0.3125	0.3125	0.3215	0.3281	0.3438	0.3281	0.3438
Iter time	0.2656	0.2344	0.8438	0.2813	0.2656	0.9375	0.6094	0.5469	1.8438
64×64	11	9	9	11	10	9	23	21	19
Setup time	3.2813	3.2813	3.2813	3.3281	3.2500	3.2813	3.3594	3.3438	3.4063
Iter time	1.3125	1.1719	4.0625	1.2969	1.2969	4.0156	2.7969	2.8438	8.2813
128×128	11	9	9	11	9	9	19	19	18
Setup time	40.3125	40.1719	40.5156	36.3125	36.1406	36.2031	36.1250	36.2656	36.1719
Iter time	9.1563	7.5625	21.6563	7.9219	6.6563	19.2813	14.5156	15.0938	42.4688

Table 2. Execution average times of single iterative process for steady Oseen problems (Q2-Q1 FEM, 64×64 uniform grid), $\nu = 0.001$ and the parameter values are respectively found experimentally.

	M-V Type	number of Nonzeros	time of M-V Multiplies	time of applying Preconditioner	time of Orthogonal.
RDF	Av	122530	0.0232	0.0903	0.0055
SPP	Av	122530	0.0253	0.1049	0.0067
MAL	$(A + B^T W^{-1} B)v$	884623	0.2574	0.0963	0.0106

preconditioned GMRES iteration counts for the leaky lid driven cavity problem discretized by Q2-Q1 and Q2-P1 finite elements. In Table 1, we use experimental optimal parameter values to obtain the corresponding iteration counts and timings. We can see that the iteration counts of the three preconditioners are more or less similar (just a difference of 1-2 iterations in many cases), thus this data is not sufficient to conclude which preconditioner is better in the case of uniform grids. However, at the same time, it is shown that the cost of the MAL preconditioner is much larger than the other two preconditioners from Table 1, and is about three times the cost of the RDF or the SPP preconditioner, here Setup time mainly shows the time of direct LU factorization of two linear subsystems.

Indeed, see [5], the RDF preconditioner can be factorized into four simple matrix factors, then it requires solving two linear systems at each iterative step. Similarly, our preconditioner can be factored as follows; see Lemma 1:

$$P = \begin{pmatrix} I & 0 & \alpha B_1^T W^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \hat{A}_1 & 0 & 0 \\ 0 & I & 0 \\ -B_1 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & \hat{A}_2 & B_2^T \\ 0 & 0 & \frac{1}{\alpha} W \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\alpha W^{-1} B_2 & I \end{pmatrix}.$$

At each step, the SPP preconditioner requires solving two linear systems with the coefficient matrices $\hat{A}_1=A_1+\alpha B_1^T W^{-1} B_1$ and $\hat{A}_2=A_2+\alpha B_2^T W^{-1} B_2$. If necessary, we also perform diagonal scaling to improve the convergence rates. Compared with the RDF preconditioner, the SPP preconditioner just additionally needs some matrix W -vector multiplications, if matrix W is simple, e.g., the diagonal matrix, then we can neglect their computation. This feature is reflected in Table 1. For the MAL preconditioner, the coefficient matrix requires more matrix-vector (M-V) multiplications obviously during iterative process, i.e., $\mathcal{O}((A+B^T W^{-1} B)v) > \mathcal{O}(Av)$, where v is a vector and \mathcal{O} indicates computational complexity. In GMRES algorithm, there are two M-V multiplication, one is initial residual obtained and the other is in the orthogonalization. Table 2 shows the average CPU times of matrix-vector multiplications, carrying out the preconditioner and orthogonalization for a single iterative process. We can find that number of nonzeros of the coefficient matrix $A + \alpha B^T W^{-1} B$ are much more, $B^T W^{-1} B$ makes sparsity of the coefficient matrix smaller. This may be more algebraic operation so that the computation time of total iteration increases. Since the linear subsystems are similar for all three preconditioners, Table 1 has shown the more or less costs for using direct LU factorization method. In Table 2, we can easily find that the coefficient matrix of the augmented system lead to the more costs.

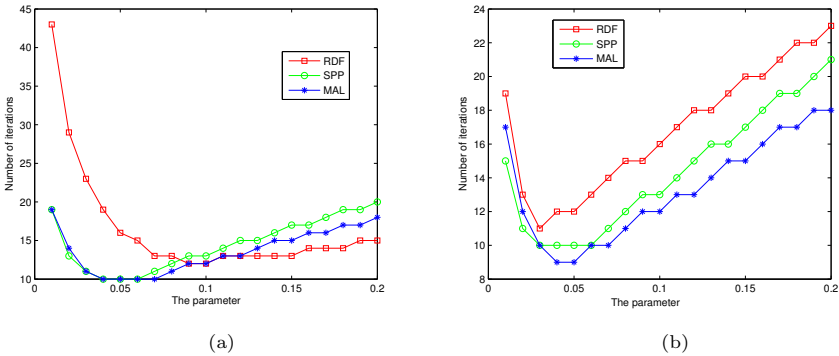


Figure 3. The sensitivity of the parameters in the RDF, SPP, and MAL preconditioners for steady Oseen problem (Q2-Q1, uniform grids, $\nu = 0.01$). (a): 32×32 , (b): 64×64 .

In Figures 3 and 4, the sensitivity of the parameter is shown for different mesh sizes and the viscosity. We can see that the sensitivity of the parameter of the MAL and the SPP preconditioners are similar, and have the same change tendency. At the same times, it's shown that the parameter of the MAL and the SPP preconditioners are robust. According to the choice of the parameter of our preconditioner and [9], in Tables 3 and 4, we know that the parameter values of the SPP preconditioner and the MAL preconditioner are equivalent, which are obtained by Fourier Analysis, i.e. $\gamma = \alpha$. For the mild viscosity, the SPP preconditioner and the MAL preconditioner are better than the RDF preconditioner, the convergence rate of GMRES of the MAL preconditioner is faster than the SPP preconditioner for the small viscosity. In Figure 5, the

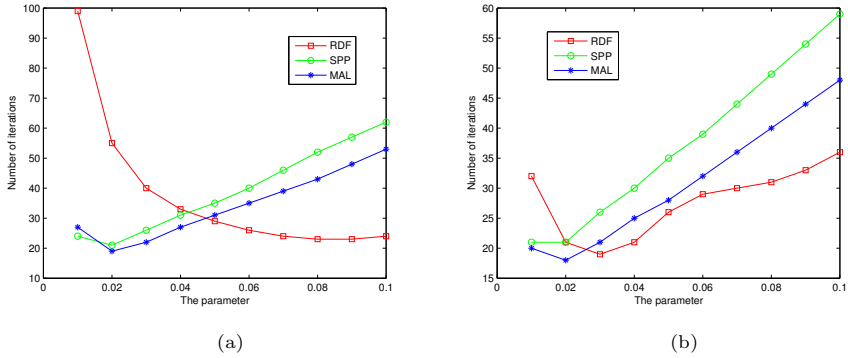


Figure 4. The sensitivity of the parameters in the RDF, SPP, and MAL preconditioners for steady Oseen problem (Q2-Q1, uniform grids, $\nu = 0.001$). (a): 64×64 . (b): 128×128 .

Table 3. Number of GMRES(20) iterations with the RDF, MAL and SPP preconditioners for steady Oseen problems (Q2-Q1 FEM uniform grids), the parameters τ , α , γ are respectively obtained by Fourier analysis.

Grid	Viscosity								
	0.1			0.01			0.001		
	RDF	SPP	MAL	RDF	SPP	MAL	RDF	SPP	MAL
16×16	11	9	9	15	10	10	52	52	47
32×32	12	9	9	15	10	10	56	37	33
64×64	12	9	9	12	10	9	38	30	26
128×128	12	9	9	11	9	10	24	21	18

Table 4. Number of GMRES(20) iterations with the RDF, MAL and SPP preconditioners for steady Oseen problems (Q2-Q1 FEM, stretched grids), the parameters τ , α , γ are respectively obtained by Fourier analysis.

Grid	Viscosity								
	0.1			0.01			0.001		
	RDF	SPP	MAL	RDF	SPP	MAL	RDF	SPP	MAL
16×16	16	9	8	17	9	9	44	35	30
32×32	32	8	9	27	10	9	67	29	23
64×64	23	9	9	25	10	9	74	26	20
128×128	16	9	9	18	9	9	58	19	14

parameters τ , α , γ obtained by Fourier Analysis and found experimentally are illustrated respectively. The parameters α and γ obtained by Fourier Analysis can work well. We can see that the SPP preconditioner has almost the same effect of the MAL preconditioner without the relatively slow convergence rate.

In Table 5, we display the number of GMRES iterations for steady problem discretized by Q2-P1 element on a stretched grid, the experimental optimal values of the parameter τ , α , γ are presented respectively in parentheses. It is shown that the RDF preconditioner converges slowly while the MAL preconditioner

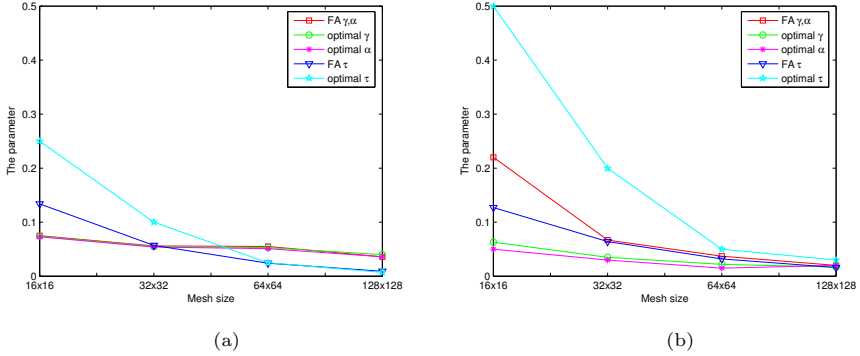


Figure 5. The parameters τ , α , γ change curves with respect to FA optimal and experimentally optimal vs. the mesh size in the RDF, SPP, and MAL preconditioners for steady Oseen problem (Q2-Q1, uniform grids). (a): $\nu = 0.01$. (b): $\nu = 0.001$.

Table 5. Number of GMRES(20) iterations with the RDF, MAL and SPP preconditioners for steady Oseen problems (Q2-P1 FEM stretched grids), the optimal parameters τ , α , γ found experimentally.

Grid	Viscosity								
	0.1			0.01			0.001		
	RDF	SPP	MAL	RDF	SPP	MAL	RDF	SPP	MAL
16 × 16	14(0.1)	8(0.3)	7(0.3)	15(0.5)	11(0.08)	10(0.08)	33(1)	22(0.03)	20(0.03)
32 × 32	18(0.02)	8(0.3)	7(0.3)	19(0.2)	11(0.08)	10(0.08)	38(0.4)	23(0.03)	20(0.03)
64 × 64	22(0.01)	10(0.3)	7(0.3)	22(0.05)	11(0.08)	10(0.08)	51(0.2)	24(0.03)	20(0.03)
128 × 128	24(0.003)	8(0.3)	7(0.3)	27(0.01)	11(0.08)	8(0.08)	55(0.04)	24(0.03)	15(0.03)

tioner and the SPP preconditioner converge quickly, and the convergence rate of the MAL preconditioner is slightly faster than that of the SPP preconditioner.

From Tables 1–5, we can see that the performance of the SPP preconditioner is good for steady problems, it has the similar convergence behaviour with the MAL preconditioner, and the similar structure with the RDF preconditioner.

Example 2. The unsteady cavity Oseen problem. Similarly, we present the unsteady problems discretized by Q2-Q1 element on a stretched grid. Linear systems of this type tend to be easier to solve than the ones arising from the steady case, since the presence of the additional positive definite term σM matrix makes the A block more diagonally dominant, where M is the velocity mass matrix; the parameter $\sigma \geq 0$ is typically proportional to the reciprocal of the time step, and is zero in the steady case. In our following experiments we also let $\sigma = 1/h$, where h is the mesh size (see [5, 7]).

In Tables 6 and 7, it’s shown that the SPP preconditioner works better than two other preconditioners. For uniform grids the parameter of the SPP preconditioner is robust and independent of the mesh size and the viscosity. In Table 8, The convergence rate for the SPP preconditioner depends on the mesh size and the viscosity, and some deterioration is observed when either decreases. The deterioration with respect to the mesh size is mild when the

Table 6. Number of GMRES(20) iterations with the RDF, MAL and SPP preconditioners for unsteady Oseen problems (Q2-Q1 FEM uniform grids). The parameters τ , α , γ are experimentally optimal.

Grid	Viscosity								
	0.1			0.01			0.001		
	RDF	SPP	MAL	RDF	SPP	MAL	RDF	SPP	MAL
16×16	10(0.01)	9(0.6)	9(0.9)	10(0.02)	9(0.6)	9(0.9)	9(0.02)	8(0.6)	8(0.6)
32×32	11(0.003)	11(0.6)	12(0.9)	11(0.006)	10(0.6)	11(0.9)	10(0.008)	9(0.6)	10(0.6)
64×64	12(0.002)	11(0.6)	13(0.9)	11(0.004)	10(0.6)	13(0.9)	9(0.001)	9(0.6)	12(0.6)
128×128	12(0.001)	11(0.6)	16(0.9)	11(0.0008)	10(0.6)	16(0.9)	12(0.0008)	9(0.6)	14(0.6)

Table 7. Number of GMRES(20) iterations with the RDF, MAL and SPP preconditioners for unsteady Oseen problems (Q2-P1 FEM uniform grids). The parameters τ , α , γ are experimentally optimal.

Grid	Viscosity								
	0.1			0.01			0.001		
	RDF	SPP	MAL	RDF	SPP	MAL	RDF	SPP	MAL
16×16	11	12	12	13	13	15	13	14	14
32×32	13	13	14	15	15	18	14	15	19
64×64	13	13	17	12	13	20	13	13	24
128×128	12	12	25	12	13	28	13	13	35

Table 8. Number of GMRES(20) iterations with the RDF, MAL and SPP preconditioners for unsteady Oseen problems (Q2-P1 FEM stretched grids). The parameters τ , α , γ are experimentally optimal.

Grid	Viscosity								
	0.1			0.01			0.001		
	RDF	SPP	MAL	RDF	SPP	MAL	RDF	SPP	MAL
16×16	14	12	13	15	17	18	16	19	20
32×32	18	15	15	19	29	32	20	31	33
64×64	23	17	18	23	30	36	25	50	57
128×128	33	18	19	28	32	40	30	62	70

viscosity is not too small, but it becomes more noticeable as ν becomes smaller. There is a clear deterioration with respect to decreasing viscosity for all h . However, these results are acceptable; see [7, 14]. The convergence rate of the SPP preconditioner is faster than the MAL preconditioner. Our preconditioner appears to be quite competitive for the unsteady problems.

Referring to [7], since the MAL preconditioner can also work well when the simple choice of the parameter $\gamma = 1$, thus in this example we also set the same parameter value in order to observe the convergence behavior of the SPP preconditioner. In Table 9, we show the rates of convergence for these preconditioners (RDF, MAL, and SPP). It is shown that the convergence rates of the RDF preconditioner are mildly dependent of mesh size, but they are independent of the viscosity when the parameter is obtained experimentally.

Table 9. Number of GMRES(20) iterations with the RDF, MAL and SPP preconditioners for unsteady Oseen problems (Q2-Q1 FEM stretched grids). The parameter τ is experimentally optimal, the parameter α, γ equal 1, the Such complement $\hat{S} = (\nu + \gamma)^{-1}W$.

Grid	Viscosity								
	0.1			0.01			0.001		
	RDF	SPP	MAL	RDF	SPP	MAL	RDF	SPP	MAL
16×16	13(0.02)	10(1)	11(1)	12(0.02)	11(1)	11(1)	12(0.02)	10(1)	11(1)
32×32	18(0.005)	12(1)	13(1)	15(0.008)	15(1)	16(1)	16(0.01)	19(1)	19(1)
64×64	21(0.0001)	14(1)	15(1)	18(0.02)	22(1)	23(1)	19(0.003)	37(1)	37(1)
128×128	23(0.0002)	17(1)	19(1)	21(0.0005)	25(1)	26(1)	25(0.0008)	55(1)	55(1)

The MAL preconditioner and the SPP preconditioner work well in the case of $\nu = 0.1, \alpha = \gamma = 1$ via using the diagonal scaling technique. However, the convergence rates of the SPP and the MAL preconditioners mildly depend on the mesh size if the parameters equal one. The smaller ν becomes, the more noticeable the deterioration gets. Nevertheless, it can be still accepted. Here we use the implicit Schur complement $\hat{S}^{-1} = (\nu + \gamma)W^{-1}$ instead of $\hat{S} = \gamma^{-1}W$ to improve the convergence behavior of GMRES for the MAL preconditioner.

From Tables 6 to 9, considering the convergence rate of preconditioner for the unsteady problems, the SPP preconditioner works better than the MAL preconditioner on uniform grids, and it is similar to the MAL preconditioner on stretched grids. If the approximate parameter values can be precise enough to tend to the experimentally optimal values, then the RDF preconditioner presents the best performance. Otherwise, the SPP preconditioner may be a nice choice for the unsteady problems.

Table 10. Number of GMRES(20) iterations with the RDF, MAL and SPP preconditioners for steady backward facing step problem (uniform grids), the iteration counts are based on the experimentally optimal parameters.

Grid	Q2-Q1			Q2-P1		
	RDF	SPP	MAL	RDF	SPP	MAL
16×48	22	18	18	23	19	19
32×96	23	18	18	23	19	19
64×192	24	18	18	25	18	18

Example 3. The backward facing step problem. We test the SPP preconditioner for the steady backward facing step problem with $\nu = 0.005$. In this example, the foregoing diagonal scaling technique is used for the RDF preconditioner. It is shown that the rates of convergence for the SPP and the MAL preconditioners are faster than the RDF preconditioner in Table 10.

6 Conclusions

In this paper, we have introduced a splitting preconditioner combining the ideas of the MAL preconditioner and the RDF preconditioner and have stud-

ied its behaviour for Navier–Stokes equations. In addition, we have compared our preconditioner with the RDF preconditioner and the MAL preconditioner, and have used Fourier analysis tool for guiding the choice of the parameter. For steady problems, our preconditioner show out the similar behaviour to the MAL preconditioner, and our preconditioner and the MAL preconditioner converge more quickly than the RDF preconditioner, especially on stretched grids. Though the convergence rate of the MAL preconditioner is slightly faster than our preconditioner, the computational cost of our preconditioner is much less than the MAL preconditioner, and is more or less compared with the RDF preconditioner. Meanwhile, Our preconditioner has the features of the RDF and the MAL preconditioners, but it more tends to be the MAL preconditioner. For unsteady case, our preconditioner works very well on uniform grids. Despite the convergence behaviour of our preconditioner mildly depends on the mesh size on stretched grids, these results are still acceptable, the performance of our preconditioner is better than the MAL preconditioner in many circumstances. All experimental results show that our preconditioner still has a certain competitiveness.

Finally, using direct methods for the solution of inner linear systems is certainly feasible, but it is not a good idea to solve larger 2D or 3D problems due to memory and time constraints, then exact solvers can be replaced with inexact solvers. Therefore, inexact solve method and 3D case need to be further considered in the future.

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