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Simultaneous Inversion of the Source Term and Initial Value of the Time Fractional Diffusion Equation

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Abstract. In this paper, the problem we investigate is to simultaneously identify the source term and initial value of the time fractional diffusion equation. This problem is ill-posed, i.e., the solution (if exists) does not depend on the measurable data. We give the conditional stability result under the *a-priori* bound assumption for the exact solution. The modified Tikhonov regularization method is used to solve this problem, and under the a-priori and the a-posteriori selection rule for the regularization parameter, the convergence error estimations for this method are obtained. Finally, numerical example is given to prove the effectiveness of this regularization method.

Keywords: time fractional diffusion equation, source term and initial value, inverse problem, ill-posed, modified Tikhonov method.

AMS Subject Classification: 35R25; 47A52; 35R30.

1 Introduction

Let Ω be the bounded area on $\mathbb{R}^d(d = 1, 2, \ldots)$. $\partial\Omega$ is the smooth boundary of Ω and $T > 0$ is a fixed time. The following time fractional diffusion equation is studied:

$$\begin{cases} \partial_t^\alpha u(x,t) = u_{xx}(x,t) + \varphi(x), & x \in \Omega,\ t \in (0,T],\ 0 < \alpha \le 1, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \\ u(x,0) = \psi(x), & x \in \Omega, \\ u(x,t_0) = f(x), & x \in \Omega,\ t_0 \in (0,T], \\ u(x,T) = g(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, $\partial_t^\alpha(\cdot)$ is the Caputo fractional derivative of the order $\alpha(0 < \alpha \le 1)$, which is defined by

$$\partial_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha} := \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha} ds, & 0 < \alpha < 1, \\ u_t(x,t), & \alpha = 1, \end{cases} \tag{1.2}$$

$\Gamma(x)$ is a Gamma function.

In problem (1.1), if $\varphi(x)$, $\psi(x)$ are known, it is a direct problem. If $\varphi(x)$, $\psi(x)$ are unknown, this is an inverse problem. In this paper, we study the inverse problem. The intermediate data $u(x,t_0) = f(x)$ and the final data $u(x,T) = g(x)$ are the given data. We use the data $f(x)$, $g(x)$ to identify the source term $\varphi(x)$ and initial value $\psi(x)$, assuming that the exact data $f(x)$, $g(x)$ and the measurement data $f^\delta(x)$, $g^\delta(x)$ satisfy

$$\|f^\delta(\cdot) - f(\cdot)\| \le \delta, \quad \|g^\delta(\cdot) - g(\cdot)\| \le \delta, \tag{1.3}$$

where $\|\cdot\|$ is the $L^2(\Omega)$ norm and $\delta > 0$ is the error level.

The inverse problem of the time fractional diffusion equation is of great interest in engineering to detect previous states of a physical field from its current information. Therefore, many researchers have done a lot of research on the inverse problem of the time fractional diffusion equation. In [22], Yang et al. used a quasi-reversibility regularization method to invert the initial value of the fractional time diffusion equation with a inhomogeneous source. Wang et al. [11] used the Tikhonov regularization method to invert the initial value of the time fractional diffusion equation over a general bounded region. Liu et al. used the strong maximum principle for fractional diffusion equations and an application to an inverse source problem [7]. In [2], Ismailov applied eigenfunction expansion method for an inverse source problem of a time-fractional diffusion equation with nonlocal boundary conditions. In [28], Zhang used the truncated method to identify an unknown source in time fractional diffusion equation. In [20], Xiong et al. made use of an optimal regularization method for an inverse heat conduction problem about a time-fractional diffusion equation. Moreover, some results concerning a homogeneous backward problem can be found in [6, 12, 14, 25].

At present, the research on the identification of source terms, initial values or diffusion coefficients of time diffusion equations has matured, and scholars have begun to explore simultaneous identification. In [21], Xiong simultaneously inverted the order of the time fractional derivative, the order of the spatial fractional derivative and the diffusion coefficient through the data at a point on the boundary under homogeneous Neumann boundary conditions. Cheng [1] et al. studied the simultaneous inversion of the order and diffusion coefficient of the time fractional diffusion equation under special initial conditions. Liao and Wei [5] used the measurement data at the endpoint to simultaneously identify the order of the fractional temporal derivative and the spatial source term. Janno [3] et al. studied the simultaneous reconstruction of the fractional order and space-dependent source term in a fractional diffusion equation from final time measured data. In [4], Li et al. used a modified optimal perturbation algorithm to deal with simultaneously recovering for the fractional order and diffusion coefficient for the time-fractional diffusion equation. Zhang [27] et al. studied the simultaneous identification of two initial values of the fractional wave equation. The topic of simultaneous identification of source item and initial value has been preliminarily studied. Jin [18] et al. used a novel modified quasi-reversibility regularization method to identify the source term and initial value of the spatial fractional diffusion equation. Qiu [9] and Jin [17, 19] et al. studied the simultaneous identification of source terms and initial values of heat equation.

However, to the best of our knowledge, there are few researches on simultaneous inversion of source term and initial value of time fractional diffusion equation. Jin [15,16] et al. studied the simultaneous identification of the source term and initial value of the time fractional diffusion equation, but did not give the error estimates. Ruan [10] et al. used the standard Tikhonov regularization method to simultaneously identify the source term and initial value of the time fractional diffusion equation. Yu [26] et al. used the exponential Tikhonov regularization method to simultaneously identify the source term and initial value of the time fractional diffusion equation. In this paper, the modified Tikhonov regularization method is used to identify simultaneously the unknown source and the initial value of the time-fractional diffusion equation. In addition, we not only obtain the priori convergence error estimate under the priori regularization parameter selection rule, but also give the posteriori convergence error estimate based on the posteriori regularization parameter selection rule.

The paper is organized as follows: Section 2 states some preparation knowledge. In Section 3, we derive the conditional stability of problem (1.1) under the a-priori bound condition for the exact solution. Section 4 uses the modified Tikhonov regularization method to solve the problem (1.1) and gives the priori and posteriori error estimates. Section 5 presents numerical example to verify the effectiveness of the regularization method.

2 Preliminary

In this section, we present some important Definitions and Lemmas.

Definition 1. Suppose $\{\lambda_n\}_{n=1}^{\infty}$, $\{X_n(x)\}_{n=1}^{\infty}$ be the Dirichlet eigenvalues and eigenfunctions of the operator $\frac{\partial^2}{\partial x^2}$ on the domain Ω:

$$\begin{cases} -\frac{\partial^2}{\partial x^2} X_n(x) = \lambda_n X_n(x), & \text{in } \Omega, \\ X_n(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots$, $\lim_{n\to\infty} \lambda_n = +\infty$ and $X_n(x) \in$

$H^2(\Omega)\cap H_0^1(\Omega)$, then $\{X_n(x)\}_{n=1}^{\infty}$ can be normalized as the orthonormal basis in space $L^2(\Omega)$.

DEFINITION 2. For any $p>0$, we define the space

$$D((-\Delta)^p)=\Big\{\phi\in L^2(\Omega)\Big|\sum_{n=1}^{\infty}\lambda_n^{2p}|(\phi,X_n)|^2<\infty\Big\},$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and $D((-\Delta)^p)$ is a Hilbert space with the norm

$$\|\phi\|_{D((-\Delta)^p)}:=\Big(\sum_{n=1}^{\infty}\lambda_n^{2p}|(\phi,X_n)|^2\Big)^{\frac{1}{2}}.$$

DEFINITION 3. [8] The Mittag-Leffler function is defined as follows:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},z\in\mathbb{C},$$

where $\alpha>0$ and $\beta\in R$ are arbitrary constants.

Lemma 1. [8] *If $\lambda>0$, then the following equality holds:*

$$\int_0^{\infty}e^{-pt}t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)}(\pm at^{\alpha})dt=\frac{m!p^{\alpha-\beta}}{(p^{\alpha}\mp a)^{m+1}},Re(p)>|a|^{\frac{1}{\alpha}},$$

where $E_{\alpha,\beta}^{(m)}(y):=\frac{d^m}{dy^m}E_{\alpha,\beta}(y)$.

Lemma 2. [23] *For $\alpha>0$ and $\gamma\in\mathbb{R}$, then*

$$E_{\alpha,\gamma}(z)=zE_{\alpha,\alpha+\gamma}(z)+1/\Gamma(\gamma),\quad z\in\mathbb{C}.$$

Lemma 3. [23] *For $0<\alpha<1,t>0$, we have $0\le E_{\alpha,1}(-t)<1$. Moreover, $E_{\alpha,1}(-t)$ is completely monotonic, that is*

$$(-1)^n\frac{d^n}{dt^n}E_{\alpha,1}(-t)\ge 0.$$

Lemma 4. [24] *Assume that $0<\alpha_0<\alpha_1<1$. Then, there exist constants $D_{\pm}>0$, depending only on α_0,α_1 such that for all $\alpha\in[\alpha_0,\alpha_1]$, we obtain*

$$\frac{D_-}{\Gamma(1-\alpha)}\frac{1}{1-x}\le E_{\alpha,1}(x)\le\frac{D_+}{\Gamma(1-\alpha)}\frac{1}{1-x},$$

for all $x\leqslant 0$.

Lemma 5. *For any λ_n that satisfies $0<\lambda_1\le\ldots\le\lambda_n$ and $\lambda_n>1$, there are normal numbers D_1, D_2 that depend on α, t_0, λ_1, and we have*

$$\frac{D_1}{\lambda_n}\le E_{\alpha,1}(-\lambda_n t_0^{\alpha})\le\frac{D_2}{\lambda_n},$$

where $D_1=\frac{D_-}{\Gamma(1-\alpha)(1+t_0^{\alpha})}$, $D_2=\frac{D_+}{\Gamma(1-\alpha)t_0^{\alpha}}$.

The proof of Lemma 5 is similar to Lemma 2.3 in [24].

Lemma 6. *For any λ_n that satisfies $0 < \lambda_1 \leq \ldots \leq \lambda_n$ and $\lambda_n > 1$, there are normal numbers D_3, D_4 that depend on α, T, λ_1, and we have*

$$D_3/\lambda_n \leq E_{\alpha,1}(-\lambda_n T^\alpha) \leq D_4/\lambda_n,$$

where $D_3 = \frac{D_-}{\Gamma(1-\alpha)(1+T^\alpha)}$, $D_4 = \frac{D_+}{\Gamma(1-\alpha)T^\alpha}$.

Lemma 7. *Assuming that $D_1 > D_4$, for any λ_n that satisfies $0 < \lambda_1 \leq \ldots \leq \lambda_n$, there is a normal number dependent on α, T, t_0, λ_1, and we have*

$$D_5/\lambda_n \leq E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(\lambda_n T^\alpha) \leq D_6/\lambda_n,$$

where $D_5 = D_1 - D_4$, $D_6 = D_2 - D_3$.

Proof. From Lemmas 5 and 6, we can obtain

$$E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(\lambda_n T^\alpha) \leq \frac{D_2}{\lambda_n} - \frac{D_3}{\lambda_n} = \frac{D_6}{\lambda_n},$$
$$E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(\lambda_n T^\alpha) \geq \frac{D_1}{\lambda_n} - \frac{D_4}{\lambda_n} = \frac{D_5}{\lambda_n},$$

where $D_5 = D_1 - D_4$, $D_6 = D_2 - D_3$. □

Lemma 8. *For any λ_n that satisfies $0 < \lambda_1 \leq \ldots \leq \lambda_n$, there exists normal numbers dependent on α, T, t_0, λ_1, and we have*

$$\frac{D_7}{\lambda_n t_0^\alpha} \leq E_{\alpha,\alpha+1}(-\lambda_n t_0^\alpha) \leq \frac{1}{\lambda_n t_0^\alpha}, \quad \frac{D_8}{\lambda_n T^\alpha} \leq E_{\alpha,\alpha+1}(-\lambda_n T^\alpha) \leq \frac{1}{\lambda_n T^\alpha},$$

where $D_7 = 1 - E_{\alpha,1}(-\lambda_1 t_0^\alpha), D_8 = 1 - E_{\alpha,1}(-\lambda_1 T^\alpha)$.

The proof of Lemma 5 is similar to Lemma 2.5 in [22].

3 The solution, the ill-posed analysis and the results of conditional stability

Using the separated variable method, the Laplace transformation and the inverse transformation of the Mittag-Leffler function, we can obtain the solution of the problem (1.1):

$$u(x,t) = \sum_{n=1}^{\infty} (\psi_n E_{\alpha,1}(-\lambda_n t^\alpha) + t^\alpha \varphi_n E_{\alpha,\alpha+1}(-\lambda_n t^\alpha)) X_n(x),$$

where $\psi_n = (\psi(x), X_n(x))$, $\varphi_n = (\varphi(x), X_n(x))$ are the Fourier coefficients. According to $u(x,t_0) = f(x)$, $u(x,T) = g(x)$ and the above formula, we can obtain $\varphi(x)$ and $\psi(x)$ corresponding expressions:

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x), \tag{3.1}$$

and

$$\psi(x)=\sum_{n=1}^{\infty}\frac{\lambda_n T^\alpha E_{\alpha,\alpha+1}(-\lambda_n T^\alpha)f_n-\lambda_n t_0^\alpha E_{\alpha,\alpha+1}(-\lambda_n t_0^\alpha)g_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}X_n(x), \quad (3.2)$$

here, $g_n=(g(x),X_n(x))$ and $f_n=(f(x),X_n(x))$ are the Fourier coefficients.

Now, we give some notations to represent (3.1) and (3.2).

$$\varphi(x)=\sum_{n=1}^{\infty}\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}X_n(x)=K_1^{-1}g+K_2^{-1}f, \quad (3.3)$$

K_1, K_2 are self-adjoint operators, and the singular values of K_1^{-1} and K_2^{-1} are as follows:

$$K_1^{-1}=\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)},\ K_2^{-1}=\frac{-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}.$$

For formula (3.2), we have the same statement as above:

$$\psi=K_3^{-1}f+K_4^{-1}g,$$

K_3, K_4 are self-adjoint operators, and the singular values of K_3^{-1} and K_4^{-1} are as follows:

$$K_3^{-1}=\frac{\lambda_n T^\alpha E_{\alpha,\alpha+1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)},\ K_4^{-1}=\frac{-\lambda_n t_0^\alpha E_{\alpha,\alpha+1}(-\lambda_n t_0^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}.$$

When $n\to\infty$, $\lambda_n\to\infty$, $K_1^{-1}\to\infty$, $K_2^{-1}\to\infty$, so from formula (3.3), the small perturbation of $g(x)$ and $f(x)$ will cause a great change in the data $\varphi(x)$. In the same way, the small perturbation of $g(x)$ and $f(x)$ will also cause a great change in the data $\psi(x)$. Therefore, this is an ill-posed problem and cannot be solved by classical methods. It needs to be solved by regularization method. Below, we give the priori bound:

$$\max\{\|\varphi(x)\|_{D((-\Delta)^p)},\ \|\psi(x)\|_{D((-\Delta)^p)}\}\le E, \quad (3.4)$$

where E, p are positive constants, $\|\varphi(x)\|_{D((-\Delta)^p)}=\left(\sum_{n=1}^{\infty}\lambda_n^{2p}|(\varphi(x),X_n)|^2\right)^{\frac{1}{2}}$, $\|\psi(x)\|_{D((-\Delta)^p)}=\left(\sum_{n=1}^{\infty}\lambda_n^{2p}|(\psi(x),X_n)|^2\right)^{\frac{1}{2}}$.

Theorem 1. *If $\varphi(x)$, $\psi(x)$ satisfy the priori bound condition* (3.4)*, then,*

$$\|\varphi(\cdot)\|\le C_1E^{\frac{2}{p+2}}(\|g\|^2+\|f\|^2)^{\frac{p}{2(p+2)}},\quad p>0,$$
$$\|\psi(\cdot)\|\le C_2E^{\frac{2}{p+2}}(\|g\|^2+\|f\|^2)^{\frac{p}{2(p+2)}},\quad p>0,$$

where $C_1=(\frac{2}{D_5})^{\frac{p}{2(p+2)}}$, $C_2=(\frac{2}{\lambda_1 D_5})^{\frac{p}{2(p+2)}}$.

Proof. Combining (3.1) with Hölder inequality, we obtain

$$\begin{aligned}
\|\varphi(\cdot)\|^2 =&\|\sum_{n=1}^{\infty}\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}X_n(x)\|^2\\
=&\sum_{n=1}^{\infty}(\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)})^2\\
=&\sum_{n=1}^{\infty}\frac{(\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^{\frac{4}{p+2}}}{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^2}\\
&\times(\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^{\frac{2p}{p+2}}\\
=&\sum_{n=1}^{\infty}\frac{(\lambda_n^{\frac{p+2}{2}}E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n^{\frac{p+2}{2}}E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^{\frac{4}{p+2}}}{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^2}\\
&\times(E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^{\frac{2p}{p+2}}\\
\le&\Big(\sum_{n=1}^{\infty}\lambda_n^p\frac{(\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^2}{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^{p+2}}\Big)^{\frac{2}{p+2}}\\
&\times\Big(\sum_{n=1}^{\infty}(E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^2\Big)^{\frac{p}{p+2}}\\
\le&\Big(\sum_{n=1}^{\infty}\frac{\lambda_n^{2p}}{D_5^p}\frac{(\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^2}{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^2}\Big)^{\frac{2}{p+2}}\\
&\times\Big(2\sum_{n=1}^{\infty}(E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n)^2+2\sum_{n=1}^{\infty}(E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^2\Big)^{\frac{p}{p+2}}\\
\le&D_5^{-\frac{p}{p+2}}\Big(\sum_{n=1}^{\infty}\lambda_n^{2p}\frac{(\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n)^2}{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^2}\Big)^{\frac{2}{p+2}}\\
&\times\Big(2\sum_{n=1}^{\infty}g_n^2+2\sum_{n=1}^{\infty}f_n^2\Big)^{\frac{p}{p+2}}\le 2^{\frac{p}{p+2}}D_5^{-\frac{p}{p+2}}E^{\frac{4}{p+2}}(\|g\|^2+\|f\|^2)^{\frac{p}{p+2}}.
\end{aligned}$$

Thus,

$$\|\varphi(\cdot)\|\le C_1E^{\frac{2}{p+2}}(\|g\|^2+\|f\|^2)^{\frac{p}{2(p+2)}},\quad C_1=\big(2/D_5\big)^{\frac{p}{2(p+2)}}.$$

The proof of $\|\psi(\cdot)\|$ is the same as that of $\|\varphi(\cdot)\|$, so it is omitted. Therefore, we complete the proof of Theorem 1. □

4 The modified Tikhonov regularization method and convergence rates

Section 3 shows that the problem (1.1) is ill-posed. If we want to restore the stability of solutions, we need to use the regularization method. In this section, we are going to use the modified Tikhonov regularization method to restore the stability of the solution. The convergence rate for the modified Tikhonov regularized solutions are obtained based on some mathematical analysis.

4.1 The modified Tikhonov regularized solutions

Consider the operator equation:

$$Kx = y,$$

where $x \in X,\ y \in Y,\ X,\ Y$ are Hilbert space, and $K : X \to Y$ is the linear bounded operator. The Tikhonov regularization method is to solve a penalized least-squares problem of the following form :

$$\min_{x \in X} \{\|Kx - y\|_Y^2 + \mu \|x\|_X^2\}.$$

By solving the penalized least-squares problem above, we can derive the minimal element $x^\mu (x \in X) : x^\mu = (K^* K + \mu I)^{-1} K^* y$, here, K^* is the adjoint operator of K.

According to the above derivation, we can get the regularization solution of the source term and the initial value after two calculations:

$$\begin{aligned} \varphi_\mu^\delta =& (K_1^* K_1 + \mu I)^{-1} K_1^* g^\delta + (K_2^* K_2 + \mu I)^{-1} K_2^* f^\delta, \\ \psi_\beta^\delta =& (K_3^* K_3 + \beta I)^{-1} K_3^* f^\delta + (K_4^* K_4 + \beta I)^{-1} K_4^* g^\delta, \end{aligned} \tag{4.1}$$

where $K_i^*,\ i = 1,2,3,4$ is the adjoint operator, and $K_i^* = K_i,\ i = 1,2,3,4$, μ and β are regularization parameters. Substitute the singular value of the operator into formula (4.1), and obtain

$$\begin{aligned} \varphi_\mu^\delta(x) =& \sum_{n=1}^{\infty} \Big(\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} k_{1n}\, g_n^\delta \\ & - \frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} k_2\, f_n^\delta \Big) X_n(x), \\ k_{1n} =& \left[1 + \mu \left(\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} \right)^2 \right]^{-1}, \\ k_{2n} =& \left[1 + \mu \left(\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} \right)^2 \right]^{-1}. \end{aligned}$$

In this paper, we use the modified kernel $1/(1 + \mu \lambda_n^2)$ instead of the kernels k_1, k_2, so the modified Tikhonov regularized solution of the source term is

$$\varphi_\mu^\delta(x) = \sum_{n=1}^{\infty} \frac{1}{1 + \mu \lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n^\delta - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n^\delta}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x). \tag{4.2}$$

In the same way, the modified Tikhonov regularized solution of the initial value is

$$\psi_\beta^\delta(x) = \sum_{n=1}^{\infty} \frac{1}{1+\beta\lambda_n^2} \frac{\lambda_n T^\alpha E_{\alpha,\alpha+1}(\lambda_n T^\alpha) f_n^\delta - \lambda_n t_0^\alpha E_{\alpha,\alpha+1}(-\lambda_n t_0^\delta) g_n^\delta}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x). \tag{4.3}$$

4.2 The priori convergence error estimates

The convergence error estimate for the modified Tikhonov regularized solution can be obtained under the priori regularization parameter choice rule, respectively.

Theorem 2. *Let $\varphi(x)$ be given by* (3.1) *and $\varphi_\mu^\delta(x)$ be given by* (4.2). *Suppose that $\varphi(x)$ satisfies the priori bound condition* (3.4)*, then we can obtain*

$$\mu = \begin{cases} (\delta/E)^{\frac{2}{p+2}}, 0 < p<2, \\ (\delta/E)^{\frac{1}{2}}, \quad p \geq 2, \end{cases} \qquad \|\varphi_\mu^\delta(x)-\varphi(x)\| \leq \begin{cases} (C_3+C_4)\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, 0<p<2, \\ (C_3 + \frac{1}{\lambda_1^{p-2}})\delta^{\frac{1}{2}} E^{\frac{1}{2}}, \ p \geq 2, \end{cases}$$

where $C_3 = 2D_5^{-1}$, $C_4 = (\frac{2-p}{p})^{\frac{2-p}{2}}/(1+\frac{2-p}{p})$.

Proof. By the triangle inequality, we have

$$\|\varphi_\mu^\delta(x) - \varphi(x)\| \leq \|\varphi_\mu^\delta(x) - \varphi_\mu(x)\| + \|\varphi_\mu(x) - \varphi(x)\|, \tag{4.4}$$

where $\varphi_\mu(x)$ is the regularized solution for noise-free data. Firstly, we give an estimate for the first term. From Lemmas 5–7, we can obtain

$$\begin{aligned}
&\|\varphi_\mu^\delta(x) - \varphi_\mu(x)\| \\
&\quad = \|\sum_{n=1}^{\infty} \frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n^\delta - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n^\delta}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x) \\
&\qquad - \sum_{n=1}^{\infty} \frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x)\| \\
&\quad = \|\sum_{n=1}^{\infty} \frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)(g_n^\delta - g_n) + \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)(f_n - f_n^\delta)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x)\| \\
&\quad \leq \|\sum_{n=1}^{\infty} \frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)(g_n^\delta - g_n)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x)\| \\
&\qquad + \|\sum_{n=1}^{\infty} \frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)(f_n - f_n^\delta)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x)\| \\
&\quad = \Big(\sum_{n=1}^{\infty} (\frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)(g_n^\delta - g_n)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2\Big)^{\frac{1}{2}} \\
&\qquad + \Big(\sum_{n=1}^{\infty} (\frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)(f_n - f_n^\delta)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2\Big)^{\frac{1}{2}} \\
&\quad \leq \sup_{n\geq 1} |\frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}|\delta \\
&\qquad + \sup_{n\geq 1} |\frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}|\delta
\end{aligned}$$

$$\leq \sup_{n\geq 1}|\frac{\lambda_n^2}{1+\mu\lambda_n^2}\frac{1}{D_5}|\delta + \sup_{n\geq 1}|\frac{\lambda_n^2}{1+\mu\lambda_n^2}\frac{1}{D_5}|\delta$$
$$\leq \sup_{n\geq 1}|\frac{\lambda_n^2}{\mu\lambda_n^2}\frac{1}{D_5}|\delta + \sup_{n\geq 1}|\frac{\lambda_n^2}{\mu\lambda_n^2}\frac{1}{D_5}|\delta = 2D_5^{-1}\mu^{-1}\delta.$$

Then,

$$\|\varphi_\mu^\delta(x) - \varphi_\mu(x)\| \leq C_3\mu^{-1}\delta, \tag{4.5}$$

where $C_3 = 2D_5^{-1}$. Next, we estimate the second term of (4.4).

$$\begin{aligned}\|\varphi_\mu(x) - \varphi(x)\| &= \|\sum_{n=1}^{\infty}\frac{1}{1+\mu\lambda_n^2}\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}X_n(x)\\
&-\sum_{n=1}^{\infty}\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}X_n(x)\|\\
&= \|\sum_{n=1}^{\infty} - \frac{\mu\lambda_n^2}{1+\mu\lambda_n^2}\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}X_n(x)\|\\
&= \Big(\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2}\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n - \lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2\Big)^{\frac{1}{2}}\\
&= \Big(\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2\lambda_n^{-2p}\lambda_n^{2p}\varphi_n^2\Big)^{\frac{1}{2}} \leq \sup_{n\geq 1}|\frac{\mu\lambda_n^2}{(1+\mu\lambda_n^2)\lambda_n^p}|\Big(\sum_{n=1}^{\infty}\lambda_n^{2p}\varphi_n^2\Big)^{\frac{1}{2}}\\
&\leq \sup_{n\geq 1}|A(n)|E,\end{aligned}$$

here $A(n)=\frac{\mu\lambda_n^2}{(1+\mu\lambda_n^2)\lambda_n^p}$. Let $s=\lambda_n$, $s > 0$, we can obtain the function $a(s) = \frac{\mu s^{2-p}}{1+\mu s^2}$. When $0 < p < 2$, because of $\lim_{s\to 0} a(s) = 0$ and $\lim_{s\to\infty} a(s) = 0$, so,

$$a(s) \leq \sup_{s\in(0,\infty)} a(s) \leq a(s_0),$$

where $s_0 \in (0,\infty)$, and s_0 satisfies $a'(s_0) = 0$. If s_0 satisfies the function $a'(s_0) = 0$, we can obtain $s_0 = ((2-p)/\mu p)^{\frac{1}{2}}$. Then,

$$a(s) \leq a_{s_0} = \frac{\mu(\frac{2-p}{\mu p})^{\frac{2-p}{2}}}{1+\mu\frac{2-p}{\mu p}} = \frac{(\frac{2-p}{p})^{\frac{2-p}{2}}}{1+\frac{2-p}{p}}\mu^{\frac{p}{2}} = C_4\mu^{\frac{p}{2}}.$$

When $p \geq 2$,

$$a(s) = \frac{\mu s^{2-p}}{1+\mu s^2} \leq \frac{\mu}{s^{p-2}} \leq \frac{1}{\lambda_1^{p-2}}\mu.$$

According to the above calculation, we can obtain

$$A(n) \leq \begin{cases} C_4\mu^{\frac{p}{2}}, & 0 < p < 2,\\ \frac{1}{\lambda_1^{p-2}}\mu, & p \geq 2.\end{cases}$$

Thus,

$$\|\varphi_\mu(x)-\varphi(x)\| \le \begin{cases} C_4\mu^{\frac{p}{2}}E, & 0<p<2,\\ \frac{1}{\lambda_1^{p-2}}\mu E, & p\ge 2. \end{cases} \tag{4.6}$$

Combining (4.5) with (4.6), we choose the regularized parameter μ by

$$\mu = \begin{cases} (\frac{\delta}{E})^{\frac{2}{p+2}}, & 0<p<2,\\ (\frac{\delta}{E})^{\frac{1}{2}}, & p\ge 2, \end{cases}$$

then, we have

$$\|\varphi_\mu^\delta(x)-\varphi(x)\| \le \begin{cases} (C_3+C_4)\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, & 0<p<2,\\ (C_3+\frac{1}{\lambda_1^{p-2}})\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & p\ge 2. \end{cases}$$

□

Theorem 3. *Let $\psi(x)$ be given by* (3.1) *and $\psi_\beta^\delta(x)$ be given by* (4.3)*. Suppose that $\psi(x)$ satisfies the priori bound condition* (3.4)*, then, we can obtain*

$$\beta = \begin{cases} (\frac{\delta}{E})^{\frac{2}{p+2}}, & 0<p<2,\\ (\frac{\delta}{E})^{\frac{1}{2}}, & p\ge 2, \end{cases}$$

$$\|\psi_\beta^\delta(x)-\psi(x)\| \le \begin{cases} (C_5+C_6)\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, & 0<p<2,\\ (C_5+1)\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & p\ge 2, \end{cases}$$

where $C_5=2/(\lambda_1 D_5)$, $C_6=(\frac{2-p}{p})^{\frac{2-p}{2}}/(1+\frac{2-p}{p})$.

The proof of Theorem 3 is similar to Theorem 2, so it is omitted.

4.3 The posteriori convergence error estimates

The priori parameter choice is based on the priori bound E of the exact solution. However, in practice the priori bound E generally can not be known easily. In this condition, we choose the regularization parameters μ, β, by adopting the posteriori rule. We consider the posteriori regularization choice rule which is called Morozov's discrepancy principal.

4.3.1 The posteriori convergence error estimates of unknown source term

Morozov's discrepancy principle for our case is to find μ such that

$$\|-\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2}(K_2g^\delta+K_1f^\delta)\| = \tau_1\delta, \tag{4.7}$$

where $\tau_1 \ge \frac{1}{D_3}+\frac{1}{D_6}$ is a constant, and $\|K_2g^\delta+K_1f^\delta\|\ge\tau_1\delta$.

Lemma 9. *Let* $\rho(\mu) = \|-\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2}(K_2 g^\delta + K_1 f^\delta)\|$*, the following results hold*

(*a*) $\rho(\mu)$ *is a continuous function*;

(*b*) $\lim_{\mu\to 0}\rho(\mu) = 0$;

(*c*) $\lim_{\mu\to\infty}\rho(\mu) = \|-(K_2 g^\delta + K_1 f^\delta)\|$;

(*d*) $\rho(\mu)$ *is a strictly increasing function over* $(0,\infty)$.

Proof. The Lemma can be easily proven with the expression

$$\rho(\mu) = \| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 (\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)} g_n^\delta + \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{-\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)} f_n^\delta) X_n(x) \|.$$

□

Remark 1. Lemma 9 indicates that there exists a unique solution for (4.7).

Lemma 10. *If* $0<\mu<1$*, we can get*

$$\mu^{-1} \le \begin{cases} (\frac{C_8}{\tau_1 - C_7})^{\frac{2}{p+2}} (\frac{E}{\delta})^{\frac{2}{p+2}}, & 0<p<2, \\ (\frac{C_9}{\tau_1 - C_7})^{\frac{1}{2}} (\frac{E}{\delta})^{\frac{1}{2}}, & p \ge 2, \end{cases}$$

where $C_7 = \frac{1}{D_3} + \frac{1}{D_1}$*,* $C_8 = \big(D_6 (\frac{2-p}{2+p})^{\frac{2-p}{4}} / (D_1^{\frac12} D_3^{\frac12}(1+\frac{2-p}{2+p}))\big)^2$*, and* $C_9 = \big(D_6/(D_1^{\frac12} D_3^{\frac12} \lambda_1^{\frac{p-2}{2}})\big)^2$.

Proof. According to (4.7) and (3.4), we have

$$\begin{aligned}
\tau_1\delta = & \| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 (\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)} g_n^\delta \\
& + \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{-\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)} f_n^\delta) X_n(x) \| \\
= & \| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)} (g_n^\delta - g_n) X_n(x) \\
& + \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)} (f_n - f_n^\delta) X_n(x) \\
& + \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 (\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)} g_n \\
& - \frac{E_{\alpha,1}(\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)} f_n) X_n(x) \| \\
\le & \| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)} (g_n^\delta - g_n) X_n(x) \|
\end{aligned}$$

$$
\begin{aligned}
&+\|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}(f_n-f_n^\delta)X_n(x)\| \\
&+\|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2(\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}g_n \\
&-\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}f_n)X_n(x)\| \\
=&\Big(\sum_{n=1}^{\infty}((\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}(g_n^\delta-g_n))^2\Big)^{\frac{1}{2}} \\
&+\Big(\sum_{n=1}^{\infty}((\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}(f_n-f_n^\delta))^2\Big)^{\frac{1}{2}} \\
&+\Big(\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^4(\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}g_n \\
&-\frac{E_{\alpha,1}(\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}f_n)^2\Big)^{\frac{1}{2}} \\
\le&\sup_{n\ge1}|(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}|\delta \\
&+\sup_{n\ge1}|(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}|\delta \\
&+\Big(\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^4(\frac{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^2}{\lambda_n^2 E_{\alpha,1}(-\lambda_n t_0^\alpha)E_{\alpha,1}(-\lambda_n T^\alpha)})^2 \\
\times&(\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}g_n-\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}f_n)^2\Big)^{\frac{1}{2}} \\
\le&(1/D_3+1/D_1)\delta \\
&+\Big(\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^4(\frac{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^2}{\lambda_n^2 E_{\alpha,1}(-\lambda_n t_0^\alpha)E_{\alpha,1}(-\lambda_n T^\alpha)})^2\lambda_n^{-2p}\lambda_n^{2p} \\
\times&(\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}g_n-\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)}f_n)^2\Big)^{\frac{1}{2}} \\
\le&C_7\delta+\sup_{n\ge1}|(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2\frac{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^2}{\lambda_n^2 E_{\alpha,1}(-\lambda_n t_0^\alpha)E_{\alpha,1}(-\lambda_n T^\alpha)}\lambda_n^{-p}| \\
&+\Big(\sum_{n=1}^{\infty}\lambda^{2p}(\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)g_n-\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha)})^2\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Thus,

$$
(\tau_1-C_7)\delta\le\sup_{n\ge1}|B(n)|^2E, \tag{4.8}
$$

where

$$
B(n)=\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2}(\frac{(E_{\alpha,1}(-\lambda_n t_0^\alpha)-E_{\alpha,1}(-\lambda_n T^\alpha))^2}{\lambda_n^2 E_{\alpha,1}(-\lambda_n t_0^\alpha)E_{\alpha,1}(-\lambda_n T^\alpha)\lambda_n^p})^{\frac{1}{2}}\le\frac{\mu\lambda_n^{\frac{2-p}{2}}}{1+\mu\lambda_n^2}\frac{D_6}{D_1^{\frac{1}{2}}D_3^{\frac{1}{2}}}.
$$

Let $s=\lambda_n$, $s>0$, we can obtain function $b(s)=\mu s^{\frac{2-p}{2}}/(1+\mu s^2)$. Similar to the calculation of $a(s)$, we get

$$b(s) \leq \begin{cases} \frac{(\frac{2-p}{2+p})^{\frac{2-p}{4}}}{1+\frac{2-p}{2+p}}\mu^{\frac{2-p}{4}}, & 0<p<2, \\ \frac{1}{\lambda_1^{\frac{p-2}{2}}}\mu, & p \geq 2, \end{cases}$$

then,

$$|B(n)|^2 \leq \begin{cases} C_8\mu^{\frac{p+2}{2}}, & 0<p<2, \\ C_9\mu^2, & p\geq 2, \end{cases}$$

where

$$C_8 = (D_6(2-p2+p)^{\frac{2-p}{4}}/D_1^{\frac{1}{2}}D_3^{\frac{1}{2}}(1+\frac{2-p}{2+p}))^2, \quad C_9 = (\frac{D_6}{D_1^{\frac{1}{2}}D_3^{\frac{1}{2}}\lambda_1^{\frac{p-2}{2}}})^2.$$

According to (4.8), we can derive that

$$\mu^{-1} \leq \begin{cases} (\frac{C_8}{\tau_1-C_7})^{\frac{2}{p+2}}(\frac{E}{\delta})^{\frac{2}{p+2}}, & 0<p<2, \\ (\frac{C_9}{\tau_1-C_7})^{\frac{1}{2}}(\frac{E}{\delta})^{\frac{1}{2}}, & p\geq 2. \end{cases}$$

□

Theorem 4. *Let $\varphi(x)$ be given by* (3.1) *and $\varphi_\mu^\delta(x)$ be given by* (4.2)*. Suppose that $\varphi(x)$ satisfies the priori bound condition* (3.4)*, and the assumptions* (1.2) *and* (1.3) *hold. The regularization parameter $\mu>0$ is chosen by the Morozov's discrepancy principle* (4.7)*.*

(1) If $0<p<2$, we can obtain the following error estimation

$$\|\varphi_\mu^\delta(x)-\varphi(x)\| \leq C_{11}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}},$$

where $C_{11}=C_3(\frac{C_8}{\tau_1-C_7})^{\frac{2}{p+2}}+C_{10}$;

(2) If $p\geq 2$, we can obtain the following error estimation

$$\|\varphi_\mu^\delta(x)-\varphi(x)\| \leq C_{12}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, p\geq 2,$$

where $C_{12}=C_3(\frac{C_9}{\tau_1-C_7})^{\frac{1}{2}}+D_5^{-\frac{1}{2}}(\frac{D_4}{\lambda_1}+\frac{D_2}{\lambda_1}+\frac{D_2D_4}{D_5}\tau)^{\frac{1}{2}}$.

Proof. When $0<p<2$, by using the triangle inequality, we have

$$\|\varphi_\mu^\delta(x)-\varphi(x)\| \leq \|\varphi_\mu^\delta(x)-\varphi_\mu(x)\|+\|\varphi_\mu(x)-\varphi(x)\|,$$

where $\varphi_\mu(x)$ is the regularized solution with no error. Firstly, we give an estimate for the first term. According to (4.5), we can obtain

$$\|\varphi_\mu^\delta(x)-\varphi_\mu(x)\| \leq C_3\mu^{-1}\delta,$$

where $C_3=2D_5^{-1}$. From Lemma 10, we have

$$\|\varphi_\mu^\delta(x)-\varphi_\mu(x)\| \leq C_3\mu^{-1}\delta \leq \begin{cases} C_3(\frac{C_8}{\tau_1-C_7})^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}E^{\frac{p}{p+2}}, & 0<p<2, \\ C_3(\frac{C_9}{\tau_1-C_7})^{\frac{1}{2}}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & p\geq 2. \end{cases} \tag{4.9}$$

Next, we estimate the second term. Denote $A_n = \frac{\mu\lambda_n^2}{1+\mu\lambda_n^2}$. For $0 < p < 2$,

$$
\begin{aligned}
&\|\varphi_\mu(x) - \varphi(x)\|^2 \\
=&\|\sum_{n=1}^{\infty} -\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x)\|^2 \\
=&\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n - \lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2 \\
=&\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^p (\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n - \lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2 (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^{2-p} \\
=&\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^p \big((E_{\alpha,1}(-\lambda_n T^\alpha) f_n - E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n)^2 (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^{2-p}\big)^{\frac{p}{p+2}} \\
&\times \big((\frac{\lambda_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^p \\
&(\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n - \lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2 (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^{2-p}\big)^{\frac{2}{p+2}} \\
\le&\Big(\sum_{n=1}^{\infty}(A_n)^{p+2} (E_{\alpha,1}(-\lambda_n T^\alpha) f_n - E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n)^2 (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^{2-p}\Big)^{\frac{p}{p+2}} \\
&\times \Big(\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^{2-p} (\frac{\lambda_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^p \\
&(\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n - \lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2\Big)^{\frac{2}{p+2}} \\
=&\Big(\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^4 (E_{\alpha,1}(-\lambda_n T^\alpha) f_n - E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n)^2\Big)^{\frac{p}{p+2}} \\
&\times \Big(\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^{2-p} (\frac{\lambda_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^p \\
&(\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n - \lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2\Big)^{\frac{2}{p+2}} \\
\le&\|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 (E_{\alpha,1}(-\lambda_n T^\alpha) f_n - E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n) X_n(x)\|^{\frac{2p}{p+2}} \\
&\times \Big(\sum_{n=1}^{\infty} D_5^{-p} \lambda_n^{2p} (\frac{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n - \lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)})^2\Big)^{\frac{2}{p+2}} \\
\le&D_5^{-\frac{2p}{p+2}} (\sum_{n=1}^{\infty} \lambda_n^{2p} \varphi_n^2)^{\frac{2}{p+2}} (\|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 E_{\alpha,1}(-\lambda_n T^\alpha)(f_n - f_n^\delta) X_n(x)\| \\
&+ \|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 E_{\alpha,1}(-\lambda_n t_0^\alpha)(g_n - g_n^\delta) X_n(x)\| \\
&+ \|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 (E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n^\delta - E_{\alpha,1}(-\lambda_n T^\alpha) f_n^\delta) X_n(x)\|)^{\frac{2p}{p+2}}
\end{aligned}
$$

$$\leq D_5^{-\frac{2p}{p+2}} E^{\frac{4}{p+2}} \Big(\big(\sum_{n=1}^{\infty} ((\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 E_{\alpha,1}(-\lambda_n T^\alpha)(f_n - f_n^\delta))^2 \big)^{\frac{1}{2}}$$
$$+ \big(\sum_{n=1}^{\infty} ((\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 E_{\alpha,1}(-\lambda_n t_0^\alpha)(g_n^\delta - g_n))^2 \big)^{\frac{1}{2}}$$
$$+ \big(\sum_{n=1}^{\infty} ((\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 (E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n^\delta - E_{\alpha,1}(-\lambda_n T^\alpha) f_n^\delta))^2 \big)^{\frac{1}{2}} \Big)^{\frac{2p}{p+2}}$$
$$\leq D_5^{-\frac{2p}{p+2}} E^{\frac{4}{p+2}} \Big(\big(\sum_{n=1}^{\infty} (\frac{D_4}{\lambda_n}(f_n - f_n^\delta))^2 \big)^{\frac{1}{2}} + \big(\sum_{n=1}^{\infty} (\frac{D_2}{\lambda_n}(g_n^\delta - g_n))^2 \big)^{\frac{1}{2}}$$
$$+ \big(\sum_{n=1}^{\infty} ((\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}$$
$$\times (\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)} g_n^\delta - \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)} f_n^\delta))^2 \big)^{\frac{1}{2}} \Big)^{\frac{2p}{p+2}}$$
$$\leq D_5^{-\frac{2p}{p+2}} E^{\frac{4}{p+2}} \Big(\sup_{n\geq 1} |\frac{D_4}{\lambda_n}|\delta + \sup_{n\geq 1} |\frac{D_2}{\lambda_n}|\delta + \sup_{n\geq 1} |\frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}|$$
$$\times \big(\sum_{n=1}^{\infty} ((\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 (\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)} g_n^\delta$$
$$- \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)} f_n^\delta))^2 \big)^{\frac{1}{2}} \Big)^{\frac{2p}{p+2}}$$
$$\leq D_5^{-\frac{2p}{p+2}} E^{\frac{4}{p+2}} \Big(\frac{D_4}{\lambda_1}\delta + \frac{D_2}{\lambda_1}\delta + \frac{D_2 D_4}{D_5}$$
$$\times \| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^2}{1+\mu\lambda_n^2})^2 (\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha)} g_n^\delta$$
$$+ \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{-\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha)} f_n^\delta)) X_n(x) \| \Big)^{\frac{2p}{p+2}}$$
$$= D_5^{-\frac{2p}{p+2}} E^{\frac{4}{p+2}} \Big(\frac{D_4}{\lambda_1}\delta + \frac{D_2}{\lambda_1}\delta + \frac{D_2 D_4}{D_5}\tau_1 \delta \Big)^{\frac{2p}{p+2}}$$
$$= D_5^{-\frac{2p}{p+2}} (\frac{D_4}{\lambda_1} + \frac{D_2}{\lambda_1} + \frac{D_2 D_4}{D_5}\tau_1)^{\frac{2p}{p+2}} \delta^{\frac{2p}{p+2}} E^{\frac{4}{p+2}}.$$

Thus,

$$\|\varphi_\mu(x) - \varphi(x)\| \leq C_{10} \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, \tag{4.10}$$

where $C_{10} = D_5^{-\frac{p}{p+2}} (\frac{D_4}{\lambda_1} + \frac{D_2}{\lambda_1} + \frac{D_2 D_4}{D_5}\tau_1)^{\frac{p}{p+2}}$.
When $p \geq 2$, according to the literature [13], it can be seen that $D((-\Delta)^p)$ is embedded in $D(-\Delta)$, we can get the conclusion.
According to (4.9) and (4.10), we can obtain

$$\|\varphi_\mu^\delta(x) - \varphi(x)\| \leq \begin{cases} C_{11} \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, & 0 < p < 2, \\ C_{12} \delta^{\frac{1}{2}} E^{\frac{1}{2}}, p \geq 2, \end{cases}$$

where $C_{11} = C_3 (\frac{C_8}{\tau_1 - C_7})^{\frac{2}{p+2}} + C_{10}$,

$C_{12} = C_3(\frac{C_9}{\tau_1 - C_7})^{\frac{1}{2}} + D_5^{-\frac{1}{2}}(\frac{D_4}{\lambda_1} + \frac{D_2}{\lambda_1} + \frac{D_2 D_4}{D_5}\tau_1)^{\frac{1}{2}}$. □

4.3.2 The posteriori convergence rates of initial value

Morozov's discrepancy principle for our case is to find β such that

$$\|\frac{\beta\lambda_n^2}{1+\beta\lambda_n^2}(K_4 g^\delta + K_3 f^\delta)\| = \tau_2\delta, \tag{4.11}$$

where $\tau_2 \geq \frac{1}{D_3} + \frac{1}{D_6}$ is a constant, and $\|K_4 g^\delta + K_3 f^\delta\| \geq \tau_2\delta$.

Lemma 11. *Let* $\rho(\beta) = \|\frac{\beta\lambda_n^2}{1+\beta\lambda_n^2}(K_4 g^\delta + K_3 f^\delta)\|$*, the following results hold*
(*a*) $\rho(\beta)$ *is a continuous function*;
(*b*) $\lim_{\beta\to 0}\rho(\beta) = 0$;
(*c*) $\lim_{\beta\to\infty}\rho(\beta) = \|K_4 g^\delta + K_3 f^\delta\|$;
(*d*) $\rho(\beta)$ *is a strictly increasing function over* $(0,\infty)$.

Proof. The Lemma can be easily proven with the expression

$$\rho(\beta) = \|\sum_{n=1}^{\infty}(\frac{\beta\lambda_n^2}{1+\beta\lambda_n^2})^2(\frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{-\lambda_n t_0^\alpha E_{\alpha,1}(-\lambda_n t_0^\alpha)} f_n^\delta + \frac{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{\lambda_n T^\alpha E_{\alpha,\alpha+1}(-\lambda_n T^\alpha)} g_n^\delta) X_n(x)\|.$$

□

Remark. Lemma 11 indicates that there exists a unique solution for (4.11).

Lemma 12. *If* $0 < \beta < 1$*, we can get*

$$\beta^{-1} \leq \begin{cases} (\frac{C_{14}}{\tau_2 - C_{13}})^{\frac{2}{p+2}}(\frac{E}{\delta})^{\frac{2}{p+2}}, & 0 < p < 2, \\ (\frac{C_{15}}{\tau_2 - C_{13}})^{\frac{1}{2}}(\frac{E}{\delta})^{\frac{1}{2}}, & p \geq 2, \end{cases}$$

where $C_{13} = 1/D_7 + 1/D_8$, $C_{14} = (D_6(\frac{2-p}{2+p})^{\frac{2-p}{4}}/D_7^{\frac{1}{2}} D_8^{\frac{1}{2}}(1 + \frac{2-p}{2+p}))^2$*, and* $C_{15} = (D_6/(D_7^{\frac{1}{2})} D_8^{\frac{1}{2}})^2$.

The proof of Lemma 12 is similar to Lemma 10, so it is omitted.

Theorem 5. *Let* $\psi(x)$ *be given by* (3.2) *and* $\psi_\beta^\delta(x)$ *be given by* (4.3)*. Suppose that* $\psi(x)$ *satisfies the priori bound condition* (3.4)*, and the assumptions* (1.2) *and* (1.3) *hold. The regularization parameter* $\beta > 0$ *is chosen by the Morozov's discrepancy principle* (4.11)*. Then,*

$$\|\psi_\beta^\delta(x) - \psi(x)\| \leq \begin{cases} C_{16}\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, & 0 < p < 2, \\ C_{17}\delta^{\frac{1}{2}} E^{\frac{1}{2}}, p \geq 2, \end{cases}$$

where $C_{16} = \frac{2}{\lambda_1 D_5}(\frac{C_{14}}{\tau_2 - C_{13}})^{\frac{2}{p+2}} + D_5^{-\frac{p}{p+2}}(\frac{2}{\lambda_1} + \frac{1}{D_5}\tau)^{\frac{p}{p+2}}$, $C_{17} = \frac{2}{\lambda_1 D_5}(\frac{C_{15}}{\tau_2 - C_{13}})^{\frac{1}{2}} + D_5^{-\frac{1}{2}}(\frac{2}{\lambda_1} + \frac{1}{D_5}\tau)^{\frac{1}{2}}$.

The proof of Theorem 5 is similar to Theorem 4, so it is omitted.

5 Numerical experiments

In this section, we present two numerical examples to illustrate the proposed method.
Step 1: We give some basic data. Let $d = 1$, $\Omega = (0, \pi)$, $t_0 = 20$, and $T = 200$. By direct calculation, we have the eigenvalues $\lambda_n = n^2$ and $X_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$ for $n = 1, 2, \ldots$. Simulation is performed using software Matlab. Additionally, we use the code Matlab $mlf(a, b, z, p)$, which was written by Igor Podlubny, to calculate the Mittag-Leffler function.
Step 2: We define

$$x_i = ih(i = 0, 1, 2, \ldots, M),\ t_n = n\iota(n = 0, 1, 2, \ldots, N),$$

where $h = \frac{1}{M}$ is the step size of spatial direction, $\iota = \frac{1}{N}$ is the step size of temporal direction.
Step 3: Example:

Example 1. Consider function

$$\varphi(x) = x(\pi - x),\ x \in [0, \pi],\ \psi(x) = x \sin(x),\ x \in [0, \pi].$$

Step 4: Direct problem to find $f(x)$, $g(x)$. For Example 1 we cannot obtain an analytical solution. At this time, we obtain the iterative format by discretizing the equation, and then obtain g, f from the iterative format. Using the discrete format of the Caputo derivative in [24], the following iterative scheme is obtained

$$\begin{cases} AU^n = -r(\sum_{n=1}^{\infty}(b_j - b_{j-1})U^{n-j} + b_{n-1}\psi_i) + \varphi_i, \\ AU^1 = \varphi_i, \end{cases}$$

here, $r = \iota^{-\alpha}/\Gamma(2-\alpha)$, $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ and

$$A = \begin{pmatrix} r + \frac{2}{h^2} & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & r + \frac{2}{h^2} & -\frac{1}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h^2} & r + \frac{1}{h^2} & -\frac{1}{h^2} \\ & & & -\frac{1}{h^2} & r + \frac{2}{h^2} \end{pmatrix}.$$

Similar to [24], take $g = U^N$, $f = U^{\frac{N+2}{10}}$. Then we get g, f.

Step 5: By adding random perturbation to noise data $f(x)$ and $g(x)$, the data with errors are obtained:

$$f^\delta = f + \varepsilon \cdot randn(size(f)),\ g^\delta = g + \varepsilon \cdot randn(size(g)),$$

where the function randn($\cdot$) produces a list of random numbers with a mean of 0 and a variance of 1, and ε represents the relative error level. Its absolute error level δ is expressed as

$$\delta_1 = \sqrt{\frac{1}{M+1}\sum_{i=1}^{M+1}(f_i - f_i^\delta)^2},\ \delta_2 = \sqrt{\frac{1}{M+1}\sum_{i=1}^{M+1}(g_i - g_i^\delta)^2},\ \delta = \frac{\delta_1 + \delta_2}{2}.$$

To verify the accuracy of the numerical results, the following relative root mean square error is defined

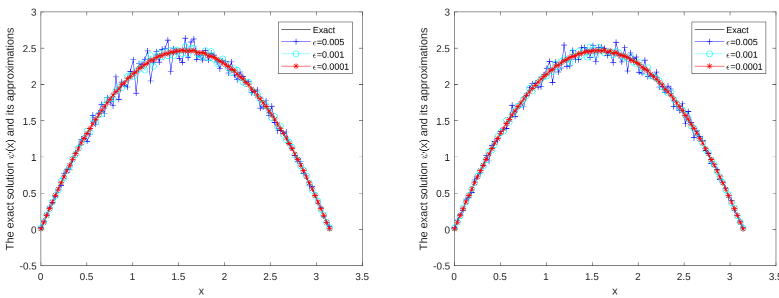
$$e_{r_1} = \frac{\sqrt{\sum(\varphi - \varphi_\mu^\delta)^2}}{\sqrt{\sum \varphi^2}}, \ e_{r_2} = \frac{\sqrt{\sum(\psi - \psi_\beta^\delta)^2}}{\sqrt{\sum \psi^2}}.$$

Step 6: The regularization results. Applying the modified Tikhonov regularization method, we have the following solutions with measurement data $f^\delta(x)$, $g^\delta(x)$:

$$\varphi_\mu^\delta(x) = \sum_{n=1}^{m} \frac{1}{1+\mu\lambda_n^2} \frac{\lambda_n E_{\alpha,1}(-\lambda_n t_0^\alpha) g_n^\delta - \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) f_n^\delta}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x),$$

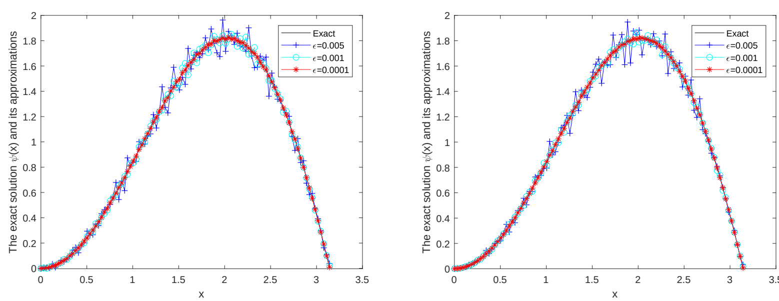
$$\psi_\beta^\delta(x) = \sum_{n=1}^{m} \frac{1}{1+\beta\lambda_n^2} \frac{\lambda_n T^\alpha E_{\alpha,\alpha+1}(\lambda_n T^\alpha) f_n^\delta - \lambda_n t_0^\alpha E_{\alpha,\alpha+1}(-\lambda_n t_0^\alpha) g_n^\delta}{E_{\alpha,1}(-\lambda_n t_0^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)} X_n(x),$$

here m is the truncation parameter and m takes 10.



(a) $\alpha = 0.2$ (b) $\alpha = 0.8$

Figure 1. The comparison of the exact solution and the regularization solution of the source term for Example 1.



(a) $\alpha = 0.2$ (b) $\alpha = 0.8$

Figure 2. The comparison of the exact solution and the regularization solution of the initial value for Example 1.

Table 1. The relative error between the exact and the regular solutions for different α.

ε			0.005	0.001	0.0001
$\varphi(x)$	$\alpha = 0.2$	e_{r_1}	0.0801	0.0574	0.0220
	$\alpha = 0.8$	er_1	0.0710	0.0457	0.0136
$\psi(x)$	$\alpha = 0.2$	e_{r_2}	0.0924	0.0542	0.0186
	$\alpha = 0.8$	e_{r_2}	0.0802	0.0339	0.0160

Figure 1 shows the comparison between the exact solution $\varphi(x)$(source term) and the regularization solution in Example 1. Figure 2 shows the comparison between the exact solution $\psi(x)$(initial value) and the regularization solution in Example 1. The numerical results of relative error at $\alpha = 0.2,\ 0.8,\ \varepsilon = 0.005,\ 0.001,\ 0.0001$ are given in Table 1. Regularization parameters are chosen as $\mu = 7.8\times10^{-4},\ 3.7\times10^{-4},\ 1.1\times10^{-4}$, $\beta = 9.4\times10^{-4},\ 5.7\times10^{-4},\ 2.1\times10^{-4}$.

From Figure 1, we can see the comparison of the source term. When $\alpha = 0.2$, the fitting effect of the graph is slightly worse than that of $\alpha = 0.8$. However, according to the data in Table 1, the relative error of $\alpha = 0.2,\ 0.8$ is not much different. And from Figure 2 and Table 1, we can also observe that in the comparison chart of the initial value, the fitting effect of $\alpha = 0.2$ is slightly worse than that of $\alpha = 0.8$, but the relative error of $\alpha = 0.2,\ 0.8$ in the Table 1 does not change much. These show that no matter how α changes in the interval $[0, 1]$, the fitting effect is relatively stable, which also mean that the modified Tikhonov regularization method we adopted is effective.

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