

On Nonhomogeneous Boundary Value Problem for the Stationary Navier-Stokes Equations in a Symmetric Cusp Domain

Kristina Kaulakytė and Neringa Kloviene

Vilnius University, Institute of Applied Mathematics
Naugarduko g. 24, LT-03225 Vilnius, Lithuania
E-mail(*corresp.*): kristina.kaulakyte@mif.vu.lt
E-mail: kloviene@gmail.com

Received February 26, 2020; revised October 22, 2020; accepted October 23, 2020

Abstract. The nonhomogeneous boundary value problem for the stationary Navier-Stokes equations in 2D symmetric multiply connected domain with a cusp point on the boundary is studied. It is assumed that there is a source or sink in the cusp point. A symmetric solenoidal extension of the boundary value satisfying the Leray-Hopf inequality is constructed. Using this extension, the nonhomogeneous boundary value problem is reduced to homogeneous one and the existence of at least one weak symmetric solution is proved. No restrictions are assumed on the size of fluxes of the boundary value.

Keywords: stationary Navier–Stokes equations, nonhomogeneous boundary condition, cusp point singularity, multiply connected domain, nonzero fluxes.

AMS Subject Classification: 35Q30; 76D03; 76D05.

1 Introduction

Mathematically a point source is a singularity from which flux or flow is emanating. Although such singularities do not exist in the observable universe, mathematical point sources/sinks are often used as approximations to reality in physics and other fields. Point sources/sinks are often used as simple models for driving flow through a gap in a wall. In oceanography, point sources are used to model the influx of fluid from channels and holes. Another example: a nuclear explosion can be treated as a thermal point source in large-scale atmospheric simulations.

This paper is devoted to the famous problem for stationary Navier-Stokes equations in the domain with multiply connected boundary. In fact, this problem arose in the pioneering Leray work (1933). Up to now, this problem is not solved for the general three dimensional case. But it has the positive solution for two dimensional (plane and axially symmetric) flows (see [7] and references there).

In this paper we consider the stationary Navier-Stokes equations with the nonhomogeneous boundary condition

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega \setminus \{O\} \end{cases} \quad (1.1)$$

in a two-dimensional multiply connected symmetric¹ cusp domain $\Omega = \Omega_0 \cup G$, where $O = (0, 0)$ is a cusp point, $\Omega_0 = \{x \in \Omega : x_2 > H\}$ and $G = \{x \in \mathbb{R}^2 : |x_1| \leq \varphi(x_2), 0 < x_2 \leq H\}$ with the function $\varphi = \varphi(x_2)$ satisfying the Lipschitz condition

$$|\varphi(t_1) - \varphi(t_2)| \leq L|t_1 - t_2|, \quad t_1, t_2 \leq H, \quad L - \text{Lipschitz constant.}$$

Moreover, $\varphi(t) > 0$ for $t > 0$ and $\varphi(t) \rightarrow 0, \varphi'(t) \rightarrow 0$ as $t \rightarrow 0$. The domain Ω_0 has the form $\Omega_0 = G_0 \setminus \cup_{i=1}^N G_i$, where G_0 and $G_i, i = 1, \dots, N$, are bounded simply connected domains such that $\overline{G_i} \subset G_0$ and $\overline{G_{i_1}} \cap \overline{G_{i_2}} = \emptyset$. Each boundary $\Gamma_i = \partial G_i, i = 1, \dots, N$, intersects the x_2 -axis at two points. The boundary $\partial\Omega$ is composed of the inner boundaries $\Gamma_1, \dots, \Gamma_N$ and the outer boundary $\partial\Omega \setminus \cup_{i=1}^N \Gamma_i = \Gamma$, i.e., the outer boundary Γ encloses the inner boundaries $\Gamma_1, \dots, \Gamma_N$ (see Figure 1).

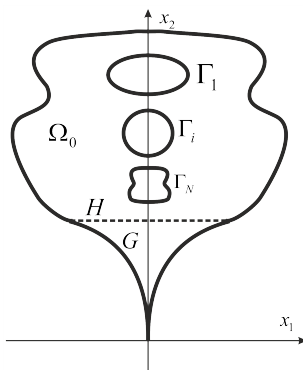


Figure 1. The domain Ω .

In (1.1) the vector-valued function $\mathbf{u} = \mathbf{u}(x)$ is the unknown velocity field, the scalar function $p = p(x)$ is the pressure of the fluid, $\nu > 0$ is the constant viscosity of the fluid, while the vector-valued function $\mathbf{a} = \mathbf{a}(x)$ denotes the given boundary value.

¹ Domain Ω is symmetric with respect to the x_2 -axis (see (2.1)).

We assume that the support of the boundary value $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ is separated from the cusp point $O = (0, 0)$, i.e.,

$$\text{supp } \mathbf{a} \subset \Lambda \cup \left(\cup_{i=1}^N \Gamma_i \right),$$

where $\Lambda \subset \Gamma \cap \partial\Omega_0$ is a connected set. Let

$$F^{(out)} = \int_{\Lambda} \mathbf{a} \cdot \mathbf{n} \, dS, \quad F_i^{(inn)} = \int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n} \, dS, \quad i = 1, \dots, N,$$

be the fluxes of the boundary value \mathbf{a} over the outer and the inner boundaries, respectively. Here \mathbf{n} is a unit vector of the outward normal to $\partial\Omega$. Since the fluid is incompressible, the total flux has to be zero (the necessary compatibility condition) and we have:

$$\int_{\sigma(R)} \mathbf{u} \cdot \mathbf{n} \, dS = -(F^{(inn)} + F^{(out)}), \quad 0 < R < H,$$

where $F^{(inn)} = \sum_{i=1}^N F_i^{(inn)}$ and $\sigma(R) = (-\varphi(R), \varphi(R))$ is a cross section of G by the straight line parallel to the x_1 -axis.

Since, the velocity \mathbf{u} has a nonzero flux in the cusp point, it has to be singular: $\mathbf{u}(x) \sim 1/\text{meas}(\sigma(r)) = c/\varphi(r) \rightarrow +\infty$ as $r \rightarrow 0$. Moreover, the velocity \mathbf{u} has infinite Dirichlet integral $\int_{\Omega} |\nabla \mathbf{u}|^2 dx = +\infty$ (infinite dissipation of energy). Therefore, it is necessary to look for the solution in a class of functions with infinite Dirichlet integral. Notice that the above formulated problem has similarities with boundary value problems for the Navier–Stokes equations in domains with paraboloidal outlets to infinity (the paraboloidal outlet to infinity in 2D has the form $\{x \in \mathbb{R}^2 : |x_1| \leq \varphi(x_2), x_2 \in (H, +\infty)\}$, where $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$). Thus, the structure of such outlet is similar to the structure of the cusp point with the difference that "singularity" is at infinity.

In multiply connected domains with outlets to infinity the solvability of non-homogeneous boundary value problem for the stationary Navier-Stokes equations is proved either assuming the smallness condition of the fluxes over the inner boundaries (see, for instance, [3,5,12,13]) or under the certain symmetry assumptions on the domain and the boundary value (see, for instance, [1,6,10,11]). In [9, 14] O.A. Ladyzhenskaya and V.A. Solonnikov proposed a method (so called Saint-Venant’s estimates method) which allowed to prove the existence of solutions having infinite Dirichlet integral in domains with outlets to infinity. In the present paper we use these ideas in the case of a source or sink in the cusp point and prove the existence of a solution to problem (1.1) for arbitrarily large fluxes $F^{(inn)}$ and $F^{(out)}$. The most important part in this prove is to construct the vector field which satisfies so called Leray-Hopf’s inequalities (see (3.4)).

2 Main notation and auxiliary results

We will use the letter “c” for a generic constant which numerical value or dependence on parameters is unessential to our considerations; “c” may have

different values in a single computation. Vector valued functions are denoted by bold letters while function spaces for scalar and vector valued functions are denoted in the same way.

Let D be a domain in \mathbb{R}^n . $C^\infty(D)$ denotes the set of all infinitely differentiable functions defined on Ω and $C_0^\infty(D)$ is the subset of all functions from $C^\infty(D)$ with compact supports in D . For given non-negative integers k and $q > 1$, $L^q(D)$ and $W^{k,q}(D)$ denote the usual Lebesgue and Sobolev spaces; $W^{k-1/q,q}(\partial D)$ is the trace space on $\partial\Omega$ of functions from $W^{k,q}(D)$. $J_0^\infty(D)$ is the set of all solenoidal ($\operatorname{div} \mathbf{u} = 0$) vector fields \mathbf{u} from $C_0^\infty(D)$.

We say that Ω is a symmetric domain with respect to the x_2 -axis if the following condition is valid:

$$(x_1, x_2) \in \Omega \Leftrightarrow (-x_1, x_2) \in \Omega. \quad (2.1)$$

The vector function $\mathbf{u} = (u_1, u_2)$ defined in Ω is called symmetric with respect to the x_2 -axis if u_1 is an odd function in x_1 and u_2 is an even function in x_1 , i.e.,

$$u_1(x_1, x_2) = -u_1(-x_1, x_2), \quad u_2(x_1, x_2) = u_2(-x_1, x_2). \quad (2.2)$$

For any set $V(\Omega)$ consisting of functions defined in the symmetric domain Ω (satisfying (2.1)), we denote by $V_S(\Omega)$ the subspace of symmetric functions (satisfying (2.2)) from $V(\Omega)$.

Let Ω be a domain with a cusp point defined in Introduction. Let us introduce a family of domains Ω_k with Lipschitz boundaries:

$$\Omega_k = \Omega_{k-1} \cup \{x \in \mathbb{R}^2 : |x_1| \leq \varphi(x_2), x_2 \in (h_k, h_{k-1})\} = \Omega_{k-1} \cup \omega_k,$$

where

$$h_0 = H, \quad h_k = h_{k-1} - \frac{\varphi(h_{k-1})}{2L}, \quad k = 1, 2, \dots$$

We write $u \in W_{loc}^{l,q}(\overline{\Omega})$ if $u \in W^{l,q}(\Omega_k)$ for $\forall k$.

Let \mathcal{M} be a closed set in \mathbb{R}^2 . By $\Delta_{\mathcal{M}}(x)$ we denote Stein's regularized distance from the point x to the set \mathcal{M} . Notice that $\Delta_{\mathcal{M}}(x)$ is an infinitely differentiable function in $\mathbb{R}^2 \setminus \mathcal{M}$ and the following inequalities

$$a_1 d_{\mathcal{M}}(x) \leq \Delta_{\mathcal{M}}(x) \leq a_2 d_{\mathcal{M}}(x), \quad |D^\alpha \Delta_{\mathcal{M}}(x)| \leq a_3 d_{\mathcal{M}}^{1-|\alpha|}(x),$$

hold, where $d_{\mathcal{M}}(x) = \operatorname{dist}(x, \mathcal{M})$ is the distance from x to \mathcal{M} . The positive constants a_1 , a_2 and a_3 are independent of \mathcal{M} (see [16], Chapter VI, Sections 1 and 2, 167-171, Theorem 2).

We shall use the well known results which are formulated in the lemmas below.

Lemma 1. *Let $D \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary ∂D , $\mathcal{L} \subseteq \partial D$, $\operatorname{meas}(\mathcal{L}) > 0$. Then for any $\mathbf{w} \in W^{1,2}(D)$ with $\mathbf{w}|_{\mathcal{L}} = 0$ the following inequality*

$$\int_D \frac{|\mathbf{w}(x)|^2 dx}{d_{\mathcal{L}}^2(x)} \leq c \int_D |\nabla \mathbf{w}|^2 dx \quad (2.3)$$

holds (see [8], Chapter V, Section 4, 129-130).

Lemma 2. Let $D \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary ∂D , $\mathcal{L} \subseteq \partial D$, $\text{meas}(\mathcal{L}) > 0$. Assume that the vector field $\mathbf{h} \in W^{1/2,2}(\partial D)$ satisfies the conditions $\int_{\mathcal{L}} \mathbf{h} \cdot \mathbf{n} dS = 0$, $\text{supp } \mathbf{h} \subseteq \mathcal{L}$. Then \mathbf{h} can be extended inside D in the form

$$\mathbf{A}(x, \varepsilon) = \left(\frac{\partial(\chi(x, \varepsilon)\mathbf{E}(x))}{\partial x_2}, -\frac{\partial(\chi(x, \varepsilon)\mathbf{E}(x))}{\partial x_1} \right),$$

where $\mathbf{E} \in W^{2,2}(D)$, $(\frac{\partial \mathbf{E}(x)}{\partial x_2}, -\frac{\partial \mathbf{E}(x)}{\partial x_1})|_{\partial D} = \mathbf{h}$ and $\chi = \chi(x, \varepsilon)$ is Hopf's type cut-off function, i.e., χ is smooth, $\chi(x, \varepsilon) = 1$ on \mathcal{L} , $\text{supp } \chi$ is contained in a small neighborhood of \mathcal{L} and

$$|\nabla \chi(x, \varepsilon)| \leq \frac{\varepsilon c}{d_{\mathcal{L}}(x)}.$$

The constant c is independent of $\varepsilon > 0$ (see [8], Chapter V, Section 4, 127-128).

3 Solvability of problem (1.1)

DEFINITION 1. A symmetric weak solution of problem (1.1) is a solenoidal vector field $\mathbf{u} \in W_{loc, S}^{1,2}(\overline{\Omega})$ satisfying the nonhomogenous boundary condition $\mathbf{u}|_{\partial\Omega \setminus \{O\}} = \mathbf{a}$ and the integral identity

$$\nu \int_{\Omega} \nabla \mathbf{u} : \nabla \boldsymbol{\eta} dx - \int_{\Omega} (\mathbf{u} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{u} dx = 0, \quad \forall \boldsymbol{\eta} \in J_{0, S}^{\infty}(\Omega). \tag{3.1}$$

Let us reduce the nonhomogeneous boundary conditions to homogeneous ones. To do this, we need to construct a suitable extension \mathbf{A} of the boundary value \mathbf{a} . Since we are looking for the symmetric solution, \mathbf{A} has to be symmetric. Moreover, it has to be solenoidal and to satisfy the condition $\mathbf{A}|_{\partial\Omega} = \mathbf{a}$. When \mathbf{A} is constructed we can look for the solution \mathbf{u} of problem (1.1) in the form

$$\mathbf{u}(x) = \mathbf{A}(x, \varepsilon) + \mathbf{v}(x), \tag{3.2}$$

where $\mathbf{v} \in W_{loc, S}^{1,2}(\overline{\Omega})$ is a new unknown solenoidal velocity field which satisfies the homogeneous boundary condition $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \setminus \{O\}$.

Putting (3.2) into (3.1) we get the integral identity for \mathbf{v} :

$$\begin{aligned} & \nu \int_{\Omega} \nabla \mathbf{v} : \nabla \boldsymbol{\eta} dx - \int_{\Omega} ((\mathbf{A} + \mathbf{v}) \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{v} dx - \int_{\Omega} (\mathbf{v} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{A} dx \\ & = \int_{\Omega} (\mathbf{A} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{A} dx - \nu \int_{\Omega} \nabla \mathbf{A} : \nabla \boldsymbol{\eta} dx, \quad \forall \boldsymbol{\eta} \in J_{0, S}^{\infty}(\Omega). \end{aligned} \tag{3.3}$$

The existence of \mathbf{v} satisfying (3.3) could be proved using the ideas proposed by O.A. Ladyzenskaya and V.A Solonnikov ([9,14]). Actually, in order to get the desirable a priori estimates, the important step is to construct the extension \mathbf{A} satisfying so called Leray-Hopf's inequalities:

$$\begin{aligned} \left| \int_{\Omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A} dx \right| & \leq \varepsilon \int_{\Omega_k} |\nabla \mathbf{w}|^2 dx, \\ \left| \int_{\omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A} dx \right| & \leq \varepsilon \int_{\omega_k} |\nabla \mathbf{w}|^2 dx, \end{aligned} \tag{3.4}$$

for any solenoidal $\mathbf{w} \in W_{loc, S}^{1,2}(\overline{\Omega})$, $\mathbf{w} = \mathbf{0}$ on $\partial\Omega \setminus \{O\}$.

As soon as we have a suitable extension \mathbf{A} of the boundary value \mathbf{a} , the method of Saint-Venant's estimates can be applied and the existence of the solution can be proved. The detailed existence proof for a simply connected cusp domain can be found in [4]. This proof remains valid for the problem considering in this paper. Therefore, we just formulate the existence theorem.

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a multiply connected symmetric with respect to the x_2 -axis cusp domain and each Γ_i , $i = 1, \dots, N$, and Γ intersect the x_2 -axis (see Figure 1). Assume that the boundary value \mathbf{a} is a symmetric vector field in $W^{1/2,2}(\partial\Omega)$ such that the support of \mathbf{a} is separated from the cusp point O . Then problem (1.1) admits at least one weak solution $\mathbf{u} = \mathbf{A} + \mathbf{v}$ which satisfies the following estimate*

$$\int_{\Omega_k} |\nabla \mathbf{u}|^2 dx \leq c \left(\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^2 + \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^4 \right) \int_{h_k}^H \frac{dx_2}{\varphi^3(x_2)},$$

where the constant c is independent of k .

4 Construction of the extension

The extension \mathbf{A} of the boundary value \mathbf{a} will be constructed as the sum

$$\mathbf{A} = \mathbf{B}^{(inn)} + \mathbf{B}^{(out)},$$

where $\mathbf{B}^{(inn)}$ extends the boundary value \mathbf{a} from the inner boundaries $\cup_{i=1}^N \Gamma_i$ and $\mathbf{B}^{(out)}$ extends the modified boundary value from the outer boundary Γ . Indeed, in order to construct $\mathbf{B}^{(inn)}$, we “remove” the fluxes $F_i^{(inn)}$, $i = 1, \dots, N$, to the outer boundary Γ . After this step we have the flux $\sum_{i=1}^N F_i^{(inn)} + F^{(out)}$ on Γ . Then by “removing” it to the cusp point and extending the modified boundary value from Γ into Ω we construct the extension $\mathbf{B}^{(out)}$. The vector fields $\mathbf{B}^{(inn)}$ and $\mathbf{B}^{(out)}$ are constructed to satisfy Leray-Hopf's inequalities which allow to obtain a priori estimates of the solutions for arbitrarily large fluxes $F_i^{(inn)}$, $i = 1, \dots, N$, and $F^{(out)}$.

Notice that the symmetry assumption is crucial for the construction of $\mathbf{B}^{(inn)}$, satisfying Leray-Hopf's inequalities. In general case Leray-Hopf's inequalities cannot be true for the vector field $\mathbf{B}^{(inn)}$. Indeed, if the fluxes over connected components of the boundary do not vanish, there is a counterexample (see [17]) showing that in general bounded domains Leray-Hopf's inequalities can be false whatever the choice of the solenoidal extension is taken. However, such extension is possible for symmetric bounded domains (see [2]).

4.1 Construction of the extension $\mathbf{B}^{(inn)}$

In order to construct $\mathbf{B}^{(inn)}$ satisfying the Leray-Hopf inequalities, we follow the idea of Fujita [2] for bounded symmetric domain.

We start with the construction of some auxiliary functions. Let $0 < \kappa < 1/2$ be a parameter. Then we introduce non-negative functions $\beta_\kappa(t) : \mathbb{R} \rightarrow \mathbb{R}$ with

the following properties:

$$\beta_\kappa(t) \in C_0^\infty(-\infty, +\infty), \beta_\kappa(-t) = \beta_\kappa(t), \forall t \in \mathbb{R}, \beta_\kappa(t) \leq \frac{1}{t} \text{ for } 0 < t < +\infty,$$

$$\beta_\kappa(t) = \begin{cases} 0, & |t| \geq 1, \\ 1/t, & \kappa \leq |t| \leq 1/2. \end{cases}$$

Define $y_\kappa = \int_{-\infty}^{+\infty} \beta_\kappa(t) dt$. Due to the properties of $\beta_\kappa(t)$, we see that

$$y_\kappa = \int_{-\infty}^{+\infty} \beta_\kappa(t) dt = \int_{-1}^1 \beta_\kappa(t) dt \geq 2 \int_\kappa^{1/2} \frac{1}{t} dt \rightarrow +\infty \text{ as } \kappa \rightarrow +0.$$

Let δ be a small positive number. Define a smooth non-negative function

$$s(t) = s(t, \delta, \kappa) = \frac{1}{y_\kappa \delta} \beta\left(\frac{t}{\delta}\right).$$

Obviously, $s(t) \in C_0^\infty(-\infty, +\infty)$ and $\text{supp } s \subseteq [-\delta, \delta]$.

Moreover,

$$\int_{-\infty}^{+\infty} s(t) dt = \int_{-\delta}^\delta s(t) dt = 1.$$

Indeed,

$$\int_{-\infty}^{+\infty} s(t) dt = \int_{-\infty}^{+\infty} \frac{1}{y_\kappa \delta} \beta\left(\frac{t}{\delta}\right) dt = \frac{1}{y_\kappa \delta} \delta \int_{-\infty}^{+\infty} \beta_\kappa\left(\frac{t}{\delta}\right) d\left(\frac{t}{\delta}\right) = \frac{1}{y_\kappa} y_\kappa = 1.$$

Furthermore,

$$0 \leq s(t) \leq \frac{1}{y_\kappa \delta} \frac{\delta}{|t|} = \frac{1}{y_\kappa |t|}, \text{ for } t \neq 0.$$

Therefore, we have that

$$\lim_{\kappa \rightarrow +0} \sup_t |t| s(t, \delta, \kappa) = 0.$$

Let us choose a small number δ so that the straight line $x_1 = \delta$ cuts each of Γ_i , $i = 1, \dots, N$ at only two points. For each Γ_i , $i = 1, \dots, N$, the x_2 -axis intersects Γ_i at the point $(0, X_i)$ and $(0, X_i^*)$, where $X_i > X_i^*$. For $i = 1, \dots, N$ we define the thin strips: $\mathcal{Y}_i = [-\delta, \delta] \times [X_i - \mu_i, X_0 + \mu_0]$, where μ_i and μ_0 are small positive numbers and $(0, X_0)$ is the point where the outer boundary Γ intersects the x_2 -axis. Notice that the points $(0, X_i - \mu_i)$ and $(0, X_0 + \mu_0)$ are outside the domain Ω and $(0, X_i - \mu_i)$ lies in G_i , $i = 1, \dots, N$ (see Figure 2).

In every strip $\mathcal{Y}_i \cap \Omega$, $i = 1, \dots, N$, we define the vector field $\mathbf{b}_i^{(inn)}$ by the formula

$$\mathbf{b}_i^{(inn)}(x) = (0, -F_i^{(inn)} s(x_1)).$$

Notice that $\mathbf{b}_i^{(inn)}$ defined on $\mathcal{Y}_i \cap \Omega$ can be extended by zero into the whole domain Ω . It is possible because the bottom of each \mathcal{Y}_i is outside the domain

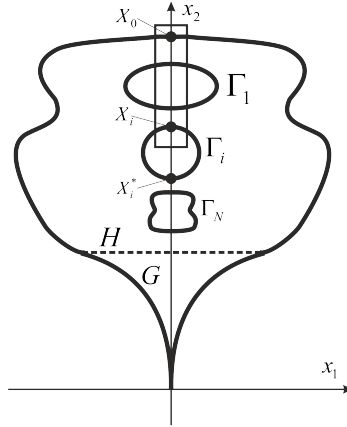


Figure 2. The strip Υ_i .

Ω (inside G_i). For the sake of simplicity we keep the same notation $\mathbf{b}_i^{(inn)}$ for the extension and set

$$\mathbf{b}_i^{(inn)}(x) = \begin{cases} (0, -F_i^{(inn)}s(x_1)) & \text{in } \Upsilon_i \cap \Omega, \\ (0, 0) & \text{in } \overline{\Omega} \setminus \Upsilon_i. \end{cases}$$

Let us take a part of $\Upsilon_i \cap \Omega$ which we denote by $\tilde{\Upsilon}_i$, i.e., the boundary of $\tilde{\Upsilon}_i$ is the union of $\Gamma_i \cap \Upsilon_i$, $[-\delta, \delta] \times (X_i + \mu)$ and the lines $x_1 = \delta$, $x_1 = -\delta$. Here μ is a small positive number. Then since the vector field $\mathbf{b}_i^{(inn)}$ is solenoidal, we obtain

$$\begin{aligned} 0 &= \int_{\tilde{\Upsilon}_i} \operatorname{div} \mathbf{b}_i^{(inn)} dx = \int_{\partial \tilde{\Upsilon}_i} \mathbf{b}_i^{(inn)} \cdot \mathbf{n} dS \\ &= \int_{\Gamma_i \cap \Upsilon_i} \mathbf{b}_i^{(inn)} \cdot \mathbf{n} dS + \int_{[-\delta, \delta] \times (X_i + \mu)} \mathbf{b}_i^{(inn)} \cdot \mathbf{e}_1 dS \\ &= \int_{\Gamma_i} \mathbf{b}_i^{(inn)} \cdot \mathbf{n} dS + \int_{-\delta}^{\delta} (0, -F_i^{(inn)}s(x_1)) \cdot (0, 1)^T dx_1 \\ &= \int_{\Gamma_i} \mathbf{b}_i^{(inn)} \cdot \mathbf{n} dS - F_i^{(inn)} \int_{-\delta}^{\delta} s(x_1) dx_1 = \int_{\Gamma_i} \mathbf{b}_i^{(inn)} \cdot \mathbf{n} dS - F_i^{(inn)}. \end{aligned}$$

Notice that the vector \mathbf{n} denotes the unit outward normal to $\partial\Omega$ on Γ_i , while the vector \mathbf{e}_1 denotes the unit normal to $\partial\tilde{\Upsilon}_i$ on $[-\delta, \delta] \times (X_i + \mu)$, vectors \mathbf{n} and \mathbf{e}_1 have opposite directions. Therefore,

$$\int_{\Gamma_i} \mathbf{b}_i^{(inn)} \cdot \mathbf{n} dS = F_i^{(inn)}.$$

Moreover,

$$\int_{\Gamma_j} \mathbf{b}_i^{(inn)} \cdot \mathbf{n} dS = \begin{cases} F_i^{(inn)}, & j = i, \\ 0, & j \neq i, j = 1, \dots, N. \end{cases}$$

Notice that for $j > i$ vector field $\mathbf{b}_i^{(inn)}$ vanishes on Γ_j (by construction) and for $j < i$ the flux of $\mathbf{b}_i^{(inn)}$ across Γ_j cancel each other.

Let us set

$$\mathbf{b}^{(inn)}(x) = \sum_{i=1}^N \mathbf{b}_i^{(inn)}(x).$$

The vector field $\mathbf{b}^{(inn)}$ is symmetric and solenoidal. Moreover, for $i = 1, \dots, N$ we have

$$\int_{\Gamma_i} (\mathbf{a} - \mathbf{b}^{(inn)}) \cdot \mathbf{n} dS = \int_{\Gamma_i} (\mathbf{a} - \mathbf{b}_i^{(inn)}) \cdot \mathbf{n} dS = F_i^{(inn)} - F_i^{(inn)} = 0. \tag{4.1}$$

Here we have used that $\mathbf{b}_i^{(inn)}$ vanishes on Γ_j for $i \neq j$.

Since condition (4.1) is valid, according to Lemma 2, there exists a solenoidal extension $\mathbf{b}_0^{(inn)}$ of $(\mathbf{a} - \mathbf{b}^{(inn)})|_{\cup_{i=1}^N \Gamma_i}$ such that the support of $\mathbf{b}_0^{(inn)}$ is contained in a small neighborhood of $\cup_{i=1}^N \Gamma_i$, $\mathbf{b}_0^{(inn)}|_{\cup_{i=1}^N \Gamma_i} = (\mathbf{a} - \mathbf{b}^{(inn)})|_{\cup_{i=1}^N \Gamma_i}$ and $\mathbf{b}_0^{(inn)}$ satisfies the Leray-Hopf inequality:

$$\left| \int_{\Omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_0^{(inn)} dx \right| \leq c\varepsilon \int_{\Omega_k} |\nabla \mathbf{w}|^2 dx,$$

where $\mathbf{w} \in W_{loc,S}^{1,2}(\overline{\Omega})$ is a solenoidal function with $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$.

Notice that $\mathbf{b}_0^{(inn)}$ is not necessary symmetric. However, since the boundary value $(\mathbf{a} - \mathbf{b}^{(inn)})|_{\cup_{i=1}^N \Gamma_i}$ is symmetric, $\mathbf{b}_0^{(inn)} = (b_{0,1}^{(inn)}, b_{0,2}^{(inn)})$ can be symmetrized. We define $\tilde{\mathbf{b}}_0^{(inn)} = (\tilde{b}_{0,1}^{(inn)}, \tilde{b}_{0,2}^{(inn)})$ by the formula

$$\begin{aligned} \tilde{b}_{0,1}^{(inn)}(x) &= \frac{1}{2} \left(b_{0,1}^{(inn)}(x_1, x_2) - b_{0,1}^{(inn)}(-x_1, x_2) \right), \quad x \in \Omega, \\ b_{0,2}^{(inn)}(x) &= \frac{1}{2} \left(b_{0,2}^{(inn)}(x_1, x_2) + b_{0,2}^{(inn)}(-x_1, x_2) \right), \quad x \in \Omega. \end{aligned} \tag{4.2}$$

Finally we define

$$\mathbf{B}^{(inn)} = \mathbf{b}^{(inn)} + \tilde{\mathbf{b}}_0^{(inn)}.$$

$\mathbf{B}^{(inn)}$ is a symmetric extension of the boundary value \mathbf{a} from $\cup_{i=1}^N \Gamma_i$. It remains to prove that $\mathbf{B}^{(inn)}$ satisfies the Leray-Hopf inequalities.

Lemma 3. *Let $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ be a symmetric vector-valued function. Then for $\forall \varepsilon > 0$ there exists a symmetric solenoidal extension $\mathbf{B}^{(inn)}$ in Ω satisfying the Leray-Hopf inequality, i.e., for every symmetric solenoidal $\mathbf{w} \in W_{loc}^{1,2}(\overline{\Omega})$ with $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$ the following estimate*

$$\left| \int_{\Omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{B}^{(inn)} dx \right| \leq c\varepsilon \int_{\Omega_k} |\nabla \mathbf{w}|^2 dx \tag{4.3}$$

holds.

Proof. Indeed, it is enough to prove that each $\mathbf{b}_i^{(inn)}$, $i = 1, \dots, N$, satisfies (4.3). To do this we use the identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)(-u_2, u_1). \quad (4.4)$$

Let $\mathbf{w} = (w_1, w_2) \in W_{loc}^{1,2}(\overline{\Omega})$, $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$, be a symmetric and solenoidal vector field. Then due to (4.4) we obtain ²:

$$\begin{aligned} & \left| \int_{\Omega_k} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \mathbf{b}_i^{(inn)} dx \right| = |F_i^{(inn)}| \int_{\mathcal{I}_i \cap \overline{\Omega}} \left| \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right| |w_1| |s(x_1)| dx \\ & = |F_i^{(inn)}| \int_{\mathcal{I}_i \cap \overline{\Omega}} \left| \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right| |w_1| \frac{1}{|x_1|} |x_1| |s(x_1)| dx \\ & \leq |F_i^{(inn)}| \sup_{x_1} (|x_1| |s(x_1)|) \int_{\mathcal{I}_i \cap \overline{\Omega}} \left| \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right| \frac{|w_1|}{|x_1|} dx \\ & \leq |F_i^{(inn)}| \sup_{x_1} (|x_1| |s(x_1)|) \left(\int_{\mathcal{I}_i \cap \overline{\Omega}} \left| \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right|^2 dx \right)^{1/2} \left(\int_{\mathcal{I}_i \cap \overline{\Omega}} \frac{|w_1|^2}{|x_1|^2} dx \right)^{1/2}. \end{aligned}$$

Notice that

$$\int_{\mathcal{I}_i \cap \overline{\Omega}} \left| \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right|^2 dx \leq 2 \int_{\Omega_k} |\nabla \mathbf{w}|^2 dx.$$

By Hardy's type inequality, since the component w_1 vanishes on $x_1 = 0$ (due to the symmetry), we get

$$\int_{\mathcal{I}_i \cap \overline{\Omega}} \frac{|w_1|^2}{|x_1|^2} dx \leq c \int_{\mathcal{I}_i \cap \overline{\Omega}} |\nabla \mathbf{w}|^2 dx \leq c \int_{\Omega_k} |\nabla \mathbf{w}|^2 dx.$$

Therefore,

$$\left| \int_{\Omega_k} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \mathbf{b}_i^{(inn)} dx \right| \leq c |F_i^{(inn)}| \sup_{x_1} (|x_1| |s(x_1)|) \int_{\Omega_k} |\nabla \mathbf{w}|^2 dx.$$

Since $\sup_{x_1} (|x_1| |s(x_1, \delta, \kappa)|) \rightarrow 0$ as $\kappa \rightarrow +0$, we can choose κ so small that $\sup_{x_1} (|x_1| |s(x_1, \delta, \kappa)|)$ is less than the given ε . Therefore,

$$\left| \int_{\Omega_k} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \mathbf{b}_i^{(inn)} dx \right| \leq c\varepsilon |F_i^{(inn)}| \int_{\Omega_k} |\nabla \mathbf{w}|^2 dx.$$

Notice that the integral over ω_k is equal to zero since $\mathbf{b}_i^{(inn)} = 0$ in ω_k . \square

4.2 Construction of the extension $\mathbf{B}^{(out)}$

After the construction of the extension $\mathbf{B}^{(inn)}$ of the boundary value \mathbf{a} from the inner boundaries $\Gamma_1, \dots, \Gamma_N$, we need to construct an extension $\mathbf{B}^{(out)}$ which extends the modified boundary value $\mathbf{a} - \mathbf{b}^{(inn)}$ from Γ .

² Since $\mathbf{b}_i^{(inn)}$ is solenoidal, it is L^2 -orthogonal to the first term of the right hand side of (4.4).

Let us define $\Omega^+ = \{x \in \Omega : x_1 > 0\}$, $\Gamma^+ = \{x \in \Gamma : x_1 > 0\}$, $\Lambda^+ = \{x \in \Lambda : x_1 > 0\}$, $G^+ = \{x \in G : x_1 > 0\}$, $\Omega_0^+ = \{x \in \Omega_0 : x_1 > 0\}$.

We start with the construction of the vector field $\mathbf{b}_+^{(out)}$ in the domain Ω^+ . Take any point $x^+ \in \Lambda^+$ and introduce a smooth simple curve $\gamma^+ = l^+ \cup \gamma_0^+$, where $l^+ = \{x_1 = 0 : 0 \leq x_2 \leq H\}$ and $\gamma_0^+ \subset \Omega_0^+$ connects the line l^+ with the point x^+ . The curve γ^+ does not intersect any inner boundary $\Gamma_1, \dots, \Gamma_N$ (see Fig.3).

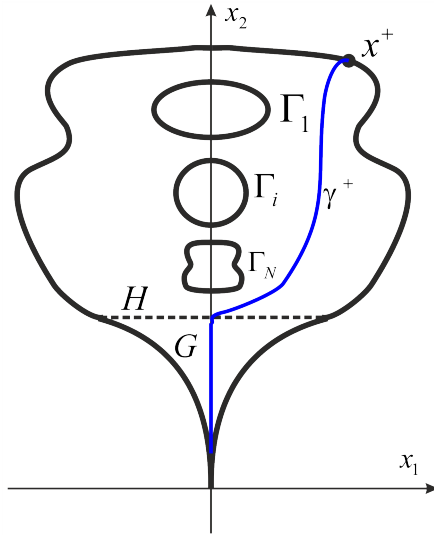


Figure 3. The curve γ^+ .

We define a cut-off function ξ^+ by the formula:

$$\xi^+(x, \varepsilon) = \Psi \left(\varepsilon \ln \frac{\Delta_{\gamma^+}(x)}{\Delta_{\partial\Omega^+ \setminus \Lambda^+}(x)} \right),$$

where Ψ is a smooth function:

$$\Psi(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1. \end{cases}$$

Lemma 4. *The function $\xi^+(x, \varepsilon)$ vanishes at those points $x \in \overline{\Omega^+} \setminus \{O\}$ where $\Delta_{\gamma^+}(x) \leq \Delta_{\partial\Omega^+ \setminus \Lambda^+}(x)$, while the curve $\gamma^+ \setminus \{O\}$ is contained in this set. The function $\xi^+(x, \varepsilon) = 1$ at points $x \in \overline{\Omega^+} \setminus \{O\}$ where $\Delta_{\partial\Omega^+ \setminus \Lambda^+}(x) \leq e^{-1/\varepsilon} \Delta_{\gamma^+}(x)$. Moreover, the following inequalities*

$$\left| \frac{\partial \xi^+(x, \varepsilon)}{\partial x_k} \right| \leq \frac{c\varepsilon}{\Delta_{\partial\Omega^+ \setminus \Lambda^+}(x)}, \quad \left| \frac{\partial^2 \xi^+(x, \varepsilon)}{\partial x_k \partial x_l} \right| \leq \frac{c\varepsilon}{\Delta_{\partial\Omega^+ \setminus \Lambda^+}^2(x)} \quad (4.5)$$

hold with the constant c independent of ε .

Proof. The first statement follows directly from the definition of the function $\Psi(t)$. Estimates (4.5) follow by direct computations using the properties of the regularized distance and the fact that $\text{supp } \nabla \xi^+$ is contained in the set where $\Delta_{\partial\Omega^+ \setminus \Lambda^+}(x) \leq \Delta_{\gamma^+}(x)$ (see for details the proof of Lemma 2 in [15]). \square

Since the curve γ^+ divides Ω^+ into two parts, we define $\tilde{\xi}^+(x, \varepsilon) = \xi^+(x, \varepsilon)$ for points laying on the right hand side of the curve γ^+ and $\xi^+(x, \varepsilon) = 0$ for points laying on the left hand side of the curve γ^+ . Then we introduce the vector field $\mathbf{b}_+^{(out)}$ in the domain Ω^+ :

$$\mathbf{b}_+^{(out)}(x, \varepsilon) = -\frac{F^{(inn)} + F^{(out)}}{2} \left(\frac{\partial \tilde{\xi}^+(x, \varepsilon)}{\partial x_2}, -\frac{\partial \tilde{\xi}^+(x, \varepsilon)}{\partial x_1} \right). \quad (4.6)$$

Lemma 5. *The vector field $\mathbf{b}_+^{(out)}$ is solenoidal, infinitely differentiable in $\overline{\Omega^+} \setminus \{O\}$, vanishes near $\partial\Omega^+ \setminus \{\Lambda^+ \cup \{O\}\}$ and on the curve $\gamma^+ \setminus \{O\}$. Moreover,*

$$\int_{\Lambda^+} \mathbf{b}_+^{(out)} \cdot \mathbf{n} dS = \frac{F^{(inn)} + F^{(out)}}{2}$$

and the following estimates

$$|\mathbf{b}_+^{(out)}(x, \varepsilon)| \leq \frac{c\varepsilon |F^{(inn)} + F^{(out)}|}{d_{\partial\Omega^+ \setminus \Lambda^+}(x)}, \quad x \in \overline{\Omega^+} \setminus \{O\}, \quad (4.7)$$

$$|\mathbf{b}_+^{(out)}(x, \varepsilon)| \leq \frac{c(\varepsilon) |F^{(inn)} + F^{(out)}|}{\varphi(x_2)}, \quad (4.8)$$

$$|\nabla \mathbf{b}_+^{(out)}(x, \varepsilon)| \leq \frac{c(\varepsilon) |F^{(inn)} + F^{(out)}|}{\varphi^2(x_2)}, \quad x \in \overline{G^+} \setminus \{O\},$$

$$|\mathbf{b}_+^{(out)}(x, \varepsilon)| + |\nabla \mathbf{b}_+^{(out)}(x, \varepsilon)| \leq c(\varepsilon) |F^{(inn)} + F^{(out)}|, \quad x \in \Omega_0^+ \quad (4.9)$$

hold with the constant c in (4.7) independent of ε .

Proof. The first statement follows directly from the definition of the vector field $\mathbf{b}_+^{(out)}$ and Lemma 4. Since $\text{div } \mathbf{b}_+^{(out)} = 0$ and due to properties of $\tilde{\xi}^+$, we have

$$\begin{aligned} \int_{\Lambda^+} \mathbf{b}_+^{(out)} \cdot \mathbf{n} dS &= - \int_{\sigma^+(R)} \mathbf{b}_+^{(out)} \cdot \mathbf{n} dS \\ &= - \int_0^{\varphi(R)} \left(-\frac{F^{(inn)} + F^{(out)}}{2} \right) \left(\frac{\partial \tilde{\xi}^+(x, \varepsilon)}{\partial x_2}, -\frac{\partial \tilde{\xi}^+(x, \varepsilon)}{\partial x_1} \right) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dx \\ &= \frac{F^{(inn)} + F^{(out)}}{2} \int_0^{\varphi(R)} \frac{\partial \tilde{\xi}^+(x, \varepsilon)}{\partial x_1} dx_1 \\ &= \frac{F^{(inn)} + F^{(out)}}{2} \left(\tilde{\xi}^+(\varphi(R), R, \varepsilon) - \tilde{\xi}^+(0, R, \varepsilon) \right) = \frac{F^{(inn)} + F^{(out)}}{2}. \end{aligned}$$

Using estimates (4.5) and definition (4.6), we derive

$$\begin{aligned} |\mathbf{b}_+^{(out)}(x, \varepsilon)| &\leq |F^{(inn)} + F^{(out)}| \left(\left(\frac{\partial \tilde{\xi}^+}{\partial x_2} \right)^2 + \left(\frac{\partial \tilde{\xi}^+}{\partial x_1} \right)^2 \right)^{1/2} \\ &\leq \frac{c\varepsilon |F^{(inn)} + F^{(out)}|}{\Delta_{\partial\Omega^+ \setminus \Lambda^+}(x)} \leq \frac{c\varepsilon |F^{(inn)} + F^{(out)}|}{d_{\partial\Omega^+ \setminus \Lambda^+}(x)} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} |\nabla \mathbf{b}_+^{(out)}(x, \varepsilon)| &\leq |F^{(inn)} + F^{(out)}| \left(\left(\frac{\partial^2 \tilde{\xi}^+}{\partial x_1 \partial x_2} \right)^2 + \left(-\frac{\partial^2 \tilde{\xi}^+}{\partial x_2 \partial x_1} \right)^2 \right)^{1/2} \\ &\leq \frac{c\varepsilon |F^{(inn)} + F^{(out)}|}{\Delta_{\partial\Omega^+ \setminus \Lambda^+}^2(x)} \leq \frac{c\varepsilon |F^{(inn)} + F^{(out)}|}{d_{\partial\Omega^+ \setminus \Lambda^+}^2(x)}. \end{aligned} \tag{4.11}$$

Since for points $x \in \text{supp } \mathbf{b}_+^{(out)}$ we have $e^{-1/\varepsilon} \Delta_{\gamma^+}(x) \leq \Delta_{\partial\Omega^+ \setminus \Lambda^+}(x) \leq \Delta_{\gamma^+}(x)$, we obtain (using the properties of the regularized distance)

$$c_1 \varphi(x_2) \leq d_{\partial\Omega^+ \setminus \Lambda^+}(x) \leq c_2 \varphi(x_2), \tag{4.12}$$

where c_1 and c_2 are positive constants. Finally, the estimates (4.8) and (4.9) follow from (4.10), (4.11) and (4.12) (see [4] for details). \square

Lemma 6. For any solenoidal $\mathbf{w} \in W_{loc}^{1,2}(\overline{\Omega^+})$ with $\mathbf{w}|_{\partial\Omega^+ \setminus \{O\}} = \mathbf{0}$ the following inequalities

$$\begin{aligned} \left| \int_{\Omega_k^+} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_+^{(out)} dx \right| &\leq \varepsilon c |F^{(inn)} + F^{(out)}| \int_{\Omega_k^+} |\nabla \mathbf{w}|^2 dx, \\ \left| \int_{\omega_k^+} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_+^{(out)} dx \right| &\leq \varepsilon c |F^{(inn)} + F^{(out)}| \int_{\omega_k^+} |\nabla \mathbf{w}|^2 dx, \end{aligned}$$

hold, where $\Omega_k^+ = \{x \in \Omega_k : x_1 > 0\}$, $\omega_k^+ = \{x \in \omega_k : x_1 > 0\}$. The constant c is independent of ε and k .

Proof. Applying the Hölder inequality and estimates (4.7), (2.3) we obtain

$$\begin{aligned} \left| \int_{\Omega_k^+} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_+^{(out)} dx \right| &\leq \left(\int_{\Omega_k^+} |\nabla \mathbf{w}|^2 dx \right)^{1/2} \left(\int_{\Omega_k^+} |\mathbf{w} \cdot \mathbf{b}_+^{(out)}|^2 dx \right)^{1/2} \\ &\leq \left(\int_{\Omega_k^+} |\nabla \mathbf{w}|^2 dx \right)^{1/2} \left(\int_{\Omega_k^+} |\mathbf{w}|^2 \left(\frac{c\varepsilon |F^{(inn)} + F^{(out)}|}{d_{\partial\Omega^+ \setminus \Lambda^+}(x)} \right)^2 dx \right)^{1/2} \\ &\leq c\varepsilon |F^{(inn)} + F^{(out)}| \int_{\Omega_k^+} |\nabla \mathbf{w}|^2 dx. \end{aligned}$$

The same argument proves the Leray-Hopf inequality in ω_k^+ . \square

Notice that Lemma 6 is also valid for the domains Ω_k^- and ω_k^- , where $\Omega_k^- = \{x \in \Omega : x_1 < 0\}$, $\omega_k^- = \{x \in \Omega : x_1 < 0\}$.

Let us extend the vector field $\mathbf{b}_+^{(out)} = (b_{+,1}^{(out)}, b_{+,2}^{(out)})$ into the domain $\Omega^- = \{x \in \Omega : x_1 < 0\}$ and define:

$$\mathbf{b}^{(out)}(x, \varepsilon) = \begin{cases} (b_{+,1}^{(out)}(x_1, x_2, \varepsilon), b_{+,2}^{(out)}(x_1, x_2, \varepsilon)), & x \in \Omega^+, \\ (-b_{+,1}^{(out)}(-x_1, x_2, \varepsilon), b_{+,2}^{(out)}(-x_1, x_2, \varepsilon)), & x \in \Omega^-. \end{cases}$$

Then $\mathbf{b}^{(out)}$ is symmetric, solenoidal, satisfies the Leray-Hopf inequalities and

$$\int_{\Lambda} \mathbf{b}^{(out)} \cdot \mathbf{n} dS = F^{(inn)} + F^{(out)}.$$

On $\partial\Omega$ we define a vector field

$$\mathbf{h}(x, \varepsilon) = \left(\mathbf{a}(x) - \mathbf{b}^{(inn)} - \mathbf{b}^{(out)} \right) \Big|_{\Lambda}, \quad \mathbf{h}(x, \varepsilon) \Big|_{\partial\Omega \setminus \Lambda} = 0.$$

Then, by construction,

$$\begin{aligned} \int_{\Lambda} \mathbf{h} \cdot \mathbf{n} dS &= \int_{\Lambda} \mathbf{a} \cdot \mathbf{n} dS - \int_{\Lambda} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS - \int_{\Lambda} \mathbf{b}^{(out)} \cdot \mathbf{n} dS = F^{(out)} \\ &- \int_{\Lambda} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS - (F^{(inn)} + F^{(out)}) = - \int_{\Lambda} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS - F^{(inn)}. \end{aligned} \quad (4.13)$$

Since $\mathbf{b}^{(inn)}$ is solenoidal, we obtain

$$\begin{aligned} 0 &= \int_{\tilde{\Upsilon}} \operatorname{div} \mathbf{b}^{(inn)} dx = \int_{\partial\tilde{\Upsilon}} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS \\ &= \int_{\Gamma \cap \Upsilon_i} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS + \int_{[-\delta, \delta] \times (X_0 - \mu)} \mathbf{b}^{(inn)} \cdot \mathbf{e}_1 dS \\ &= \int_{\Lambda} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS + \int_{-\delta}^{\delta} (0, -F^{(inn)} s(x_1)) \cdot (0, -1)^T dx_1 \\ &= \int_{\Lambda} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS + F^{(inn)} \int_{-\delta}^{\delta} s(x_1) dx_1 = \int_{\Lambda} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS + F^{(inn)}, \end{aligned}$$

where $\tilde{\Upsilon}$ is a part of $\Upsilon_i \cap \Omega$, i.e., the boundary of $\tilde{\Upsilon}$ is the union of $\Gamma \cap \Upsilon_i$, $[-\delta, \delta] \times (X_0 - \mu)$ and the lines $x_1 = \delta$, $x_1 = -\delta$.

Notice that the vector \mathbf{n} denotes the unit outward normal to $\partial\Omega$ on Γ , while the vector \mathbf{e}_1 denotes the unit normal to $\partial\tilde{\Upsilon}$ on $[-\delta, \delta] \times (X_0 - \mu)$, vectors \mathbf{n} and \mathbf{e}_1 have the opposite directions. Therefore,

$$\int_{\Lambda} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS = -F^{(inn)}. \quad (4.14)$$

From (4.13) and (4.14) we have that

$$\int_{\Lambda} \mathbf{h} \cdot \mathbf{n} dS = - \int_{\Lambda} \mathbf{b}^{(inn)} \cdot \mathbf{n} dS - F^{(inn)} = 0.$$

Because of $\int_A \mathbf{h} \cdot \mathbf{n} dS = 0$, according to Lemma 2, there exists an extension $\mathbf{b}_0^{(out)}$ of \mathbf{h} such that $\text{supp } \mathbf{b}_0^{(out)}(x, \varepsilon)$ is contained in a small neighborhood of A , $\text{div } \mathbf{b}_0^{(out)} = 0$ and $\mathbf{b}_0^{(out)}(x, \varepsilon)|_A = \mathbf{h}(x, \varepsilon)$. Moreover, $\mathbf{b}_0^{(out)}$ satisfies the Leray-Hopf inequality

$$\left| \int_{\Omega_0} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_0^{(out)} dx \right| \leq \varepsilon c |F^{(inn)} + F^{(out)}| \int_{\Omega_0} |\nabla \mathbf{w}|^2 dx. \tag{4.15}$$

However, $\mathbf{b}_0^{(out)}$ is not necessary symmetric, but since the boundary value \mathbf{h} is symmetric, vector field $\mathbf{b}_0^{(out)}$ can be symmetrized as in (4.2). Denote the symmetrized vector field by $\tilde{\mathbf{b}}_0^{(out)} = (\tilde{b}_{0,1}^{(out)}, \tilde{b}_{0,2}^{(out)})$. Finally, we put

$$\mathbf{B}^{(out)}(x, \varepsilon) = \mathbf{b}^{(out)}(x, \varepsilon) + \tilde{\mathbf{b}}_0^{(out)}(x, \varepsilon).$$

Lemma 7. *The vector field $\mathbf{B}^{(out)} \in W_{loc}^{1,2}(\bar{\Omega})$ is symmetric and solenoidal in $\bar{\Omega} \setminus \{O\}$, $\mathbf{B}^{(out)}|_A = \mathbf{a}|_A - \mathbf{b}^{(inn)}|_A$, $\mathbf{B}^{(out)}|_{\partial\Omega \setminus (A \cup \{O\})} = \mathbf{0}$. For any solenoidal symmetric vector field $\mathbf{w} \in W_{loc}^{1,2}(\bar{\Omega})$ with $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$ the following inequalities*

$$\left| \int_{\Omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{B}^{(out)} dx \right| \leq \varepsilon c |F^{(inn)} + F^{(out)}| \int_{\Omega_k} |\nabla \mathbf{w}|^2 dx, \tag{4.16}$$

$$\left| \int_{\omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{B}^{(out)} dx \right| \leq \varepsilon c |F^{(inn)} + F^{(out)}| \int_{\omega_k} |\nabla \mathbf{w}|^2 dx \tag{4.17}$$

hold with the constant c independent of ε and k . Moreover,

$$|\mathbf{B}^{(out)}(x, \varepsilon)| + |\nabla \mathbf{B}^{(out)}(x, \varepsilon)| \leq c(\varepsilon) |F^{(inn)} + F^{(out)}|, \quad x \in \Omega_0, \tag{4.18}$$

$$|\mathbf{B}^{(out)}(x, \varepsilon)| \leq \frac{c(\varepsilon) |F^{(inn)} + F^{(out)}|}{\varphi(x_2)}, \tag{4.19}$$

$$|\nabla \mathbf{B}^{(out)}(x, \varepsilon)| \leq \frac{c(\varepsilon) |F^{(inn)} + F^{(out)}|}{\varphi^2(x_2)}, \quad x \in \bar{G} \setminus \{O\}$$

$$\|\nabla \mathbf{B}^{(out)}\|_{L_2(\Omega_k)}^2 \leq c \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^2 \int_{h_k}^H \frac{dx_2}{\varphi^3(x_2)}, \tag{4.20}$$

$$\|\mathbf{B}^{(out)}\|_{L_4(\Omega_k)}^4 \leq c \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^4 \int_{h_k}^H \frac{dx_2}{\varphi^3(x_2)}, \tag{4.21}$$

where the constants in (4.20) and (4.21) are independent of k .

Proof. Estimates (4.18), (4.19), (4.16), (4.17) follow from Lemma 5, Lemma 6 and the inequality (4.15). According to the fact that

$$|F^{(inn)} + F^{(out)}| \leq c \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^2,$$

we derive estimate (4.20):

$$\begin{aligned} \|\nabla \mathbf{B}^{(out)}\|_{L_2(\Omega_k)}^2 &\leq \int_{\Omega_0} |\nabla \mathbf{B}^{(out)}|^2 dx + \int_{\Omega_k \setminus \Omega_0} |\nabla \mathbf{B}^{(out)}|^2 dx \\ &\leq \int_{\Omega_0} c |F^{(inn)} + F^{(out)}|^2 dx + \int_{\Omega_k \setminus \Omega_0} \frac{c |F^{(inn)} + F^{(out)}|^2 dx}{\varphi^4(x_2)} \end{aligned}$$

$$\begin{aligned}
&= c|F^{(inn)} + F^{(out)}|^2 \left(\text{meas}(\Omega_0) + \int_{h_k}^H \frac{dx_2}{\varphi^4(x_2)} \int_{-\varphi(x_2)}^{\varphi(x_2)} dx_1 \right) \\
&\leq c|F^{(inn)} + F^{(out)}|^2 \left(1 + \int_{h_k}^H \frac{dx_2}{\varphi^3(x_2)} \right) \\
&\leq c\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^2 \int_{h_k}^H \frac{dx_2}{\varphi^3(x_2)}.
\end{aligned}$$

Analogously we get the estimate (4.21). \square

According to Lemma 3 and Lemma 7, the constructed vector field

$$\mathbf{A} = \mathbf{B}^{(inn)} + \mathbf{B}^{(out)}$$

has all the necessary properties that insure the validity of Theorem 1 formulated in Section 3.

Acknowledgements

The research was funded by a grant No. S-MIP-17-68 from the Research Council of Lithuania.

References

- [1] M. Chipot, K. Kaulakytė, K. Pileckas and W. Xue. On nonhomogeneous boundary value problems for the stationary Navier–Stokes equations in two-dimensional symmetric semi-infinite outlets. *Analysis and Applications*, **15**(4):543–569, 2017. <https://doi.org/10.1142/S0219530515500268>.
- [2] H. Fujita. On stationary solutions to Navier–Stokes equation in symmetric plane domain under general outflow condition. In *Pitman research notes in mathematics*, volume 388 of *Proceeding of international conference on Navier-Stokes equations. Theory and numerical methods*, pp. 16–30, Varenna, Italy, 1997.
- [3] K. Kaulakytė. On nonhomogeneous boundary value problem for the steady Navier–Stokes system in domain with paraboloidal and layer type outlets to infinity. *Topol. Methods Nonlinear Anal.*, **46**(2):835–865, 2015. <https://doi.org/10.12775/TMNA.2015.070>.
- [4] K. Kaulakytė, N. Kloviėnė and K. Pileckas. Nonhomogeneous boundary value problem for the stationary Navier–Stokes equations in a domain with a cusp. *Zeitschrift für angewandte Mathematik und Physik*, **70**(36), 2019. <https://doi.org/10.1007/s00033-019-1075-5>.
- [5] K. Kaulakytė and K. Pileckas. On the nonhomogeneous boundary value problem for the Navier–Stokes system in a class of unbounded domains. *Journal of Mathematical Fluid Mechanics*, **14**(4):693–716, 2012. <https://doi.org/10.1007/s00021-011-0089-3>.
- [6] K. Kaulakytė and W. Xue. Nonhomogeneous boundary value problem for Navier–Stokes equations in 2D symmetric unbounded domains. *Applicable Analysis*, **96**(11):1906–1927, 2017. <https://doi.org/10.1080/00036811.2016.1198780>.

- [7] M.V. Korobkov, K. Pileckas, V.V. Pukhnachev and R. Russo. The flux problem for the Navier–Stokes equations. *Russian Mathematical Surveys*, **69**(6):1065–1122, 2014. <https://doi.org/10.1070/RM2014v069n06ABEH004928>.
- [8] O.A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, New York, 1969.
- [9] O.A. Ladyzhenskaya and V.A. Solonnikov. Determination of the solutions of boundary value problems for stationary Stokes and Navier–Stokes equations having an unbounded Dirichlet integral. *Journal of Soviet Mathematics*, **21**:728–761, 1983. <https://doi.org/10.1007/BF01094437>.
- [10] H. Morimoto and H. Fujita. A remark on the existence of steady Navier–Stokes flows in 2D semi-infinite channel involving the general outflow condition. *Mathematica Bohemica*, **126**(2):457–468, 2001. <https://doi.org/10.21136/MB.2001.134017>.
- [11] H. Morimoto and H. Fujita. A remark on the existence of steady Navier–Stokes flows in a certain two-dimensional infinite channel. *Tokyo Journal of Mathematics*, **25**(2):307–321, 2002. <https://doi.org/10.3836/tjm/1244208856>.
- [12] J. Neustupa. On the steady Navier–Stokes boundary value problem in an unbounded 2D domain with arbitrary fluxes through the components of the boundary. *Annali Dell'Università Di Ferrara*, **55**(2):353–365, 2009. <https://doi.org/10.1007/s11565-009-0083-3>.
- [13] J. Neustupa. A new approach to the existence of weak solutions of the steady Navier–Stokes system with inhomogeneous boundary data in domains with noncompact boundaries. *Archive for Rational Mechanics and Analysis*, **198**(1):331–348, 2010. <https://doi.org/10.1007/s00205-010-0297-7>.
- [14] V.A. Solonnikov. On the solvability of boundary and initial-boundary value problems for the Navier–Stokes system in domains with noncompact boundaries. *Pacific J. Math.*, **93**(2):443–458, 1981. <https://doi.org/10.2140/pjm.1981.93.443>.
- [15] V.A. Solonnikov and K. Pileckas. Certain spaces of solenoidal vectors and the solvability of the boundary problem for the Navier–Stokes system of equations in domains with noncompact boundaries. *Journal of Soviet Mathematics*, **34**:2101–2111, 1986. <https://doi.org/10.1007/BF01741584>.
- [16] E.M. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, 1970.
- [17] A. Takeshita. A remark on Leray’s inequality. *Pacific Journal of Mathematics*, **157**(1):151–158, 1993. <https://doi.org/10.2140/pjm.1993.157.151>.