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On a Class of Saddle Point Problems and Convergence Results

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Abstract. We consider an abstract mixed variational problem consisting of two inequalities. The first one is governed by a functional ϕ, possibly non-differentiable. The second inequality is governed by a nonlinear term depending on a non negative parameter ϵ. We study the existence and the uniqueness of the solution by means of the saddle point theory. In addition to existence and uniqueness results, we deliver convergence results for $\epsilon \to 0$. Finally, we illustrate the abstract results by means of two examples arising from contact mechanics.

Keywords: mixed variational problem, penalty term, saddle point, convergence result.

AMS Subject Classification: 47J30; 49J40; 74M10; 74M15.

1 Introduction

In the present paper we study the following variational problem.

Problem 1. Given $f, h \in X$ and $\epsilon \geq 0$, find $u_\epsilon \in K \subseteq X$ and $\lambda_\epsilon \in \Lambda \subseteq Y$ such that

$$a(u_\epsilon, v-u_\epsilon)+\phi(v)-\phi(u_\epsilon)+b(v-u_\epsilon, \lambda_\epsilon) \geq (f, v-u_\epsilon)_X \quad \text{for all}\, v \in K, \tag{1.1}$$

$$b(u_\epsilon, \mu-\lambda_\epsilon) - \epsilon(\|\mu\|_Y^2 - \|\lambda_\epsilon\|_Y^2) \leq b(h, \mu-\lambda_\epsilon) \qquad \text{for all}\ \mu \in \Lambda, \tag{1.2}$$

where

(h_1) X and Y are two Hilbert spaces;

(h_2) The form $a : X \times X \to \mathbb{R}$ is symmetric, bilinear, continuous (of rank $M_a > 0$) and $X-$elliptic (of rank $m_a > 0$);

(h_3) The form $b : X \times Y \to \mathbb{R}$ is bilinear, continuous (of rank $M_b > 0$);

(h_4) The functional $\phi : X \to \mathbb{R}_+$ is Lipschitz continuous (of rank $L_\phi > 0$) and convex;

(h_5) Λ is a closed, convex subset of Y that contains 0_Y;

(h_6) K is a closed, convex subset of X that contains 0_X.

We can associate to Problem 1 the following functional:

$$\mathcal{L}_\epsilon : K \times \Lambda \to \mathbb{R},\ \ \mathcal{L}_\epsilon(v, \mu) = \frac{1}{2}a(v, v) + b(v - h, \mu) - (f, v)_X - \epsilon\|\mu\|_Y^2 + \phi(v).$$

Assuming that Problem 1 has a solution $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$, then this solution is a saddle point of the functional $\mathcal{L}_\epsilon$. Conversely, assuming that the functional $\mathcal{L}_\epsilon$ has a saddle point $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$, then this saddle point verifies Problem 1. Thus, Problem 1 can be called *saddle point problem*.

The present study is motivated by mechanical and numerical reasons. If $\epsilon > 0$, Problem 1 is a saddle point problem with penalty term. Saddle point problems with penalty term can arise in elasticity theory, see, e.g., [3], pages 137–138. If $\epsilon = 0$, Problem 1 can be seen as a generalization of Problem (S) in [3], page 129. Also, if $\epsilon = 0$, Problem 1 can be seen as a mixed variational formulation with Lagrange multipliers for a class of contact problems; see, e.g., [11]. Due to the interest on the mixed variational formulations via Lagrange multipliers in contact mechanics, a lot of work has been done in the last decade. For recent related papers we refer to, e.g., [2, 7, 12, 13, 14, 15, 18].

Notice that $\mathcal{L}_\epsilon$ is strictly convex in the first argument and concave in the second one. If $\epsilon > 0$, $\mathcal{L}_\epsilon$ is also strictly concave in the second argument; algorithms of type multi-level can be envisaged in this case in order to approximate the solution $(u_\epsilon, \lambda_\epsilon)$.

The present work focuses on existence and uniqueness results as well as on the convergence of the sequence $(u_\epsilon, \lambda_\epsilon)_\epsilon$ when $\epsilon \to 0$. Then, two examples are delivered. Both examples are related to the weak solvability via mixed variational formulations with Lagrange multipliers of a contact model with two-contact zones, by considering a deformable body in unilateral frictionless contact on a part of the boundary and in bilateral frictional contact on another one. The "differentiable case" is related to the description of the friction by means of a regularized friction law. To simplify the presentation, the examples we deliver in the present work are only "inspired" from realistic models. By using some specific function spaces in 3D contact mechanics, other examples, more realistic models from the physical point of view, can be delivered.

The reader can consult [9, 10] for helpful techniques in the saddle point theory. However, for the convenience of the reader, we recall here two main tools: the definition of the saddle point and an existence theorem.

Definition 1. Let A and B be two non-empty sets. A pair $(u, \lambda) \in A \times B$ is said to be a saddle point of a functional $\mathcal{L} : A \times B \to \mathbb{R}$ if and only if

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda) \quad \text{for all } v \in A,\ \mu \in B.$$

Theorem 1. *Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$, $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ be two Hilbert spaces and let $A \subseteq X$, $B \subseteq Y$ be non-empty, closed, convex subsets. Assume that a functional $\mathcal{L} : A \times B \to \mathbb{R}$ satisfies the following conditions:*

$$v \to \mathcal{L}(v,\mu) \text{ is convex and lower semi-continuous for all } \mu \in B,$$
$$\mu \to \mathcal{L}(v,\mu) \text{ is concave and upper semi-continuous for all } v \in A.$$

Moreover,

$$A \text{ is bounded or } \lim_{\|v\|_X\to\infty,\, v\in A} \mathcal{L}(v,\mu_*) = \infty \text{ for some } \mu_* \in B,$$
$$B \text{ is bounded or } \lim_{\|\mu\|_Y\to\infty,\, \mu\in B} \inf_{v\in A} \mathcal{L}(v,\mu) = -\infty.$$

Then, the functional $\mathcal{L}$ has at least one saddle point.

The proof of Theorem 1 can be found in [9].

The rest of the paper has the following structure. In Section 2 we prove the existence of at least one solution $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ assuming that the functional ϕ is non-differentiable. The uniqueness in the first argument is also obtained. Furthermore, we give some convergence results for $\epsilon \to 0$. In Section 3 the study is devoted to the case when ϕ is a Gâteaux differentiable functional. This additional hypothesis allows us to obtain uniqueness as well as strong convergence in the second component of the pair solution. In Section 4 we illustrate the abstract results through two examples related to contact models involving multi-contact zones.

2 The non-differentiable case

In this section we deliver existence, uniqueness and convergence results under the hypotheses (h_1)–(h_6), assuming that the functional ϕ is non-differentiable. Let us start with an auxiliary result.

Lemma 1. *If $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ is a solution of Problem 1, then this pair is a saddle point of the functional $\mathcal{L}_\epsilon : K \times \Lambda \to \mathbb{R}$,*

$$\mathcal{L}_\epsilon(v,\mu) = \frac{1}{2}a(v,v) + b(v-h,\mu) - (f,v)_X - \epsilon\|\mu\|_Y^2 + \phi(v).$$

Conversely, assuming that the functional $\mathcal{L}_\epsilon$ has a saddle point $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$, then this pair is a solution of Problem 1.

Proof. Let $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ be a solution of Problem 1. By summing (1.2) with $\frac{1}{2}a(u_\epsilon,u_\epsilon) + \phi(u_\epsilon) - (f,u_\epsilon)_X$, we obtain

$$\mathcal{L}_\epsilon(u_\epsilon,\mu) \le \mathcal{L}_\epsilon(u_\epsilon,\lambda_\epsilon) \quad \text{for all } \mu \in \Lambda.$$

On the other hand,

$$\mathcal{L}_\epsilon(u_\epsilon,\lambda_\epsilon) - \mathcal{L}_\epsilon(v,\lambda_\epsilon) \le -\frac{1}{2}a(u_\epsilon - v, u_\epsilon - v) \le 0 \quad \text{for all } v \in K.$$

Therefore, $(u_\epsilon, \lambda_\epsilon)$ is a saddle point of $\mathcal{L}_\epsilon$.

Let us assume now that $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ is a saddle point of the functional $\mathcal{L}_\epsilon$. Since

$$\mathcal{L}_\epsilon(u_\epsilon, \mu) \leq \mathcal{L}_\epsilon(u_\epsilon, \lambda_\epsilon) \text{ for all } \mu \in \Lambda,$$

by the definition of $\mathcal{L}_\epsilon$ we immediately get (1.2). In addition, because

$$\mathcal{L}_\epsilon(u_\epsilon, \lambda_\epsilon) - \mathcal{L}_\epsilon(w, \lambda_\epsilon) \leq 0 \quad \text{for all } w \in K,$$

then,

$$\frac{1}{2}a(u_\epsilon, u_\epsilon) - \frac{1}{2}a(w, w) + \phi(u_\epsilon) - \phi(w) + b(u_\epsilon - w, \lambda_\epsilon) + (f, w - u_\epsilon)_X \leq 0.$$

Setting $w = u_\epsilon + t(v - u_\epsilon)$, with $t \in (0, 1]$ and $v \in K$,

$$t\, a(u_\epsilon, v - u_\epsilon) + \frac{t^2}{2}a(v - u_\epsilon, v - u_\epsilon) + t\,(\phi(v) - \phi(u_\epsilon)) + t\, b(v - u_\epsilon, \lambda_\epsilon) \geq t(f, v - u_\epsilon)_X.$$

Dividing by $t > 0$ and then passing to the limit as $t \to 0$, we obtain (1.1). □

Remark 1. The hypothesis (h_3) allows us to write:

for each sequence $(u_n)_n \subset X$ such that $u_n \rightharpoonup u$ in X as $n \to \infty$, we have $b(u_n, \mu) \to b(u, \mu)$ for all $\mu \in \Lambda$;

for each sequence $(\lambda_n)_n \subset Y$ such that $\lambda_n \rightharpoonup \lambda$ in Y as $n \to \infty$, we have $b(v, \lambda_n) \to b(v, \lambda)$ for all $v \in X$.

Theorem 2. *Under the hypotheses (h_1)–(h_6), if $\Lambda \subseteq Y$ is bounded, then Problem 1 has at least one solution $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$, unique in its first argument.*

Proof. In order to obtain the existence part, we consider two cases.

1. $K \subseteq X$ is a bounded subset.

By Theorem 1, we immediately deduce that the functional $\mathcal{L}_\epsilon$ has at least one saddle point $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$. So, Problem 1 has at least one solution.

2. $K \subseteq X$ is an unbounded subset.

We have to verify that

$$\lim_{\|v\|_X \to \infty, v \in K} \mathcal{L}_\epsilon(v, \mu_*) = \infty \quad \text{for some } \mu_* \in \Lambda. \tag{2.1}$$

Indeed, let $\mu_* = 0_Y$. We write,

$$\mathcal{L}_\epsilon(v, 0_Y) = \frac{1}{2}a(v, v) - (f, v)_X + \phi(v) \text{ for all } v \in K.$$

Thus,

$$\mathcal{L}_\epsilon(v, 0_Y) \geq \frac{m_a}{2}\|v\|_X^2 - \|f\|_X\, \|v\|_X \text{ for all } v \in K.$$

Passing to the limit as $\|v\|_X \to \infty$, we get (2.1). Using Theorem 1 and Lemma 1 we obtain the existence of a solution $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ of Problem 1 in this case too.

Next, we study the uniqueness. Let $(u_\epsilon^1, \lambda_\epsilon^1), (u_\epsilon^2, \lambda_\epsilon^2) \in K \times \Lambda$ be two solutions of Problem 1. For all $v \in K$, $\mu \in \Lambda$ and $i \in \{1, 2\}$,

$$a(u_\epsilon^i, v - u_\epsilon^i) + \phi(v) - \phi(u_\epsilon^i) + b(v - u_\epsilon^i, \lambda_\epsilon^i) \geq (f, v - u_\epsilon^i)_X$$
$$b(u_\epsilon^i, \mu - \lambda_\epsilon^i) - \epsilon \|\mu\|_Y^2 + \epsilon \|\lambda_\epsilon^i\|_Y^2 \leq b(h, \mu - \lambda_\epsilon^i).$$

We take $v = u_\epsilon^2,\ \mu = \lambda_\epsilon^2$, if $i = 1$ and $v = u_\epsilon^1,\ \mu = \lambda_\epsilon^1$, if $i = 2$ to obtain,

$$a(u_\epsilon^1 - u_\epsilon^2, u_\epsilon^2 - u_\epsilon^1) + b(u_\epsilon^1 - u_\epsilon^2, \lambda_\epsilon^2 - \lambda_\epsilon^1) \geq 0, \ \ b(u_\epsilon^1 - u_\epsilon^2, \lambda_\epsilon^2 - \lambda_\epsilon^1) \quad \leq 0.$$

Consequently,

$$m_a \|u_\epsilon^1 - u_\epsilon^2\|_X^2 \leq 0,$$

which implies $u_\epsilon^1 = u_\epsilon^2$. □

Below we will need an additional assumption.

(h_7) There exists $\alpha > 0$ s.t. $\inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \dfrac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$

Theorem 3. *Under the hypotheses* (h_1)–(h_7), *if* $K \subseteq X$ *is a linear subspace, then Problem 1 has a solution, unique in its first argument. Moreover,*

$$\|u_\epsilon\|_X \leq k_1, \tag{2.2}$$
$$\|\lambda_\epsilon\|_Y \leq \frac{1}{\alpha} (\|f\|_X + M_a k_1 + L_\phi), \tag{2.3}$$

where

$$k_1 = \frac{2}{m_a} \sqrt{\left(1 + \frac{m_a^2}{4 M_a^2}\right) (\|f\|_X^2 + L_\phi^2) + \frac{3 M_a^2 M_b^2}{\alpha^2} \|h\|_X^2}.$$

Proof. Let us consider the following two cases.

1. $\Lambda \subseteq Y$ is a bounded subset. As in Theorem 2 we obtain the existence of a solution $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ of Problem 1 and the uniqueness of the first argument $u_\epsilon \in K$.

2. $\Lambda \subseteq Y$ is an unbounded subset. We check if

$$\lim_{\|\mu\|_Y \to \infty, \mu \in \Lambda} \inf_{v \in K} \mathcal{L}_\epsilon(v, \mu) = -\infty. \tag{2.4}$$

Let $\mu \in \Lambda$ and let $u_\mu \in K$ be the unique solution of the variational inequality of the second kind,

$$a(u_\mu, v - u_\mu) + \phi(v) - \phi(u_\mu) \geq (f_\mu, v - u_\mu)_X \quad \text{for all } v \in K, \tag{2.5}$$

where $f_\mu \in X$ is defined by Riesz's representation theorem as follows,

$$(f_\mu, v)_X = (f, v)_X - b(v, \mu) \quad \text{for all } v \in X. \tag{2.6}$$

Since u_μ minimizes the functional

$$X \ni v \to \frac{1}{2} a(v, v) + \phi(v) - (f_\mu, v)_X,$$

then,

$$\inf_{v\in K}\mathcal{L}_\epsilon(v,\mu) \quad = \quad \frac{1}{2}a(u_\mu,u_\mu)+\phi(u_\mu)-(f,u_\mu)_X+b(u_\mu-h,\mu)-\epsilon\,\|\mu\|_Y^2.$$

Taking $v=0_X$ in (2.5) and then summing with $\frac{1}{2}a(u_\mu,u_\mu)$, we are lead to

$$\frac{1}{2}a(u_\mu,u_\mu)-(f,u_\mu)_X+\phi(u_\mu)+b(u_\mu,\mu)\le -\frac{m_a}{2}\,\|u_\mu\|_X^2+\phi(0_X).$$

Therefore,

$$\inf_{v\in K}\mathcal{L}_\epsilon(v,\mu) \quad \le \quad -\frac{m_a}{2}\|u_\mu\|_X^2-\epsilon\,\|\mu\|_Y^2-b(h,\mu)+\phi(0_X).$$

Setting now $v=u_\mu-w$ with $w\in K$, in (2.5), and keeping in mind (2.6), we obtain

$$b(w,\mu)\le (f,w)_X-a(u_\mu,w)+\phi(u_\mu-w)-\phi(u_\mu).$$

Due to the inf-sup property of the form b, we get

$$\alpha\|\mu\|_Y\le \|f\|_X+M_a\|u_\mu\|_X+L_\phi.$$

Thus,

$$\|\mu\|_Y^2\le c(\|f\|_X^2+\|u_\mu\|_X^2+L_\phi^2),$$

where $c>0$ depends on the positive constant α from (h_7).

Consequently, we can write,

$$\inf_{v\in K}\mathcal{L}_\epsilon(v,\mu)\le -\tilde{c}(\|\mu\|_Y^2-\|f\|_X^2-L_\phi^2)+M_b\,\|h\|_X\,\|\mu\|_Y-\epsilon\,\|\mu\|_Y^2+\phi(0_X),$$

where $\tilde{c}(\alpha,m_a,M_a)>0$. We pass to the limit as $\|\mu\|_Y\to\infty$ in this last relation to obtain (2.4). By Theorem 1 we get the existence of a saddle point $(u_\epsilon,\lambda_\epsilon)\in K\times\Lambda$ of the functional $\mathcal{L}_\epsilon$ and then, by Lemma 1, we deduce that this saddle point verifies Problem 1.

Furthermore, as in the previous theorem we obtain the uniqueness of the first component of the pair solution of Problem 1, $u_\epsilon\in K$.

Let us prove now (2.2) and (2.3). To this end in view, we take $v=0_X$ in (1.1) and $\mu=0_Y$ in (1.2). Hence,

$$a(u_\epsilon,u_\epsilon)\le (f,u_\epsilon)_X-b(u_\epsilon,\lambda_\epsilon)+\phi(0_X)-\phi(u_\epsilon),$$
$$-b(u_\epsilon,\lambda_\epsilon)\le -b(h,\lambda_\epsilon)-\epsilon\,\|\lambda_\epsilon\|_Y^2\le -b(h,\lambda_\epsilon).$$

Combining these last relations, we get

$$a(u_\epsilon,u_\epsilon)\le (f,u_\epsilon)_X-b(h,\lambda_\epsilon)+\phi(0_X)-\phi(u_\epsilon).$$

Thus,

$$m_a\,\|u_\epsilon\|_X^2\le \|f\|_X\,\|u_\epsilon\|_X+M_b\,\|h\|_X\,\|\lambda_\epsilon\|_Y+L_\phi\,\|u_\epsilon\|_X.$$

Consequently,

$$m_a \|u_\epsilon\|_X^2 \leq \frac{1}{2\,p_1}\|f\|_X^2 + \frac{p_1}{2}\|u_\epsilon\|_X^2 + \frac{M_b^2}{2\,p_2}\|h\|_X^2 + \frac{p_2}{2}\|\lambda_\epsilon\|_Y^2 + \frac{L_\phi^2}{2\,p_3} + \frac{p_3}{2}\|u_\epsilon\|_X^2, \tag{2.7}$$

where p_1, p_2, $p_3 > 0$. Setting now $v = u_\epsilon - w$ with $w \in K$, in (1.1), we obtain

$$b(w, \lambda_\epsilon) \leq (f, w)_X - a(u_\epsilon, w) + \phi(u_\epsilon - w) - \phi(u_\epsilon).$$

By the inf-sup property of the form b, we get

$$\|\lambda_\epsilon\|_Y \leq \frac{1}{\alpha}(\|f\|_X + M_a\|u_\epsilon\|_X + L_\phi). \tag{2.8}$$

Thus,

$$\|\lambda_\epsilon\|_Y^2 \leq \frac{3}{\alpha^2}(\|f\|_X^2 + M_a^2\|u_\epsilon\|_X^2 + L_\phi^2). \tag{2.9}$$

Combining (2.7) with (2.9) and taking $p_1 = p_3 = \frac{m_a}{2}$, $p_2 = \frac{m_a\,\alpha^2}{6\,M_a^2}$, we obtain

$$\frac{m_a}{4}\|u_\epsilon\|_X^2 \leq \left(\frac{1}{m_a} + \frac{m_a}{4\,M_a^2}\right)(\|f\|_X^2 + L_\phi^2) + \frac{3\,M_a^2\,M_b^2}{m_a\,\alpha^2}\|h\|_X^2.$$

Therefore,

$$\|u_\epsilon\|_X^2 \leq \frac{4}{m_a^2}\left[\left(1 + \frac{m_a^2}{4\,M_a^2}\right)(\|f\|_X^2 + L_\phi^2) + \frac{3\,M_a^2\,M_b^2}{\alpha^2}\|h\|_X^2\right]. \tag{2.10}$$

By (2.10) and (2.8) we immediately get (2.2) and (2.3). □

Let us draw the attention to the case $\epsilon = 0$. Problem 1 leads us to the following particular problem.

Problem 2. For given $f, h \in X$, find $u_0 \in K \subseteq X$ and $\lambda_0 \in \Lambda \subseteq Y$ such that, for all $v \in K$ and $\mu \in \Lambda$ we have,

$$a(u_0, v - u_0) + \phi(v) - \phi(u_0) + b(v - u_0, \lambda_0) \geq (f, v - u_0)_X, \tag{2.11}$$

$$b(u_0, \mu - \lambda_0) \leq b(h, \mu - \lambda_0). \tag{2.12}$$

If $K = X$, then this problem was already studied, see, e.g., [5].

Let us introduce a new hypothesis as follows.

(h_8) If $(u_n)_n \subset X$ such that $u_n \to u$ in X as $n \to \infty$ and $(\lambda_n)_n \subset Y$ such that $\lambda_n \rightharpoonup \lambda$ in Y as $n \to \infty$, then $b(u_n, \lambda_n) \to b(u, \lambda)$ as $n \to \infty$.

Theorem 4. *The hypotheses (h_1)–(h_8) hold true and, in addition, we assume that $K \subseteq X$ is a linear subspace. Let $((u_\epsilon, \lambda_\epsilon))_{\epsilon>0} \subset K \times \Lambda$, where for each $\epsilon > 0$, $(u_\epsilon, \lambda_\epsilon)$ is a solution of Problem 1. Then, there exists a subsequence $((u_{\epsilon'}, \lambda_{\epsilon'}))_{\epsilon'} \subset K \times \Lambda$ and there exists $\lambda_0 \in \Lambda$ such that $u_{\epsilon'} \to u_0$ and $\lambda_{\epsilon'} \rightharpoonup \lambda_0$, as $\epsilon' \to 0$, $(u_0, \lambda_0) \in K \times \Lambda$ being a solution of Problem 2 (u_0 is the unique first component of the pair solution of Problem 2) .*

Proof. Let $\epsilon > 0$, let $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ be a solution of Problem 1 and let $(u_0, \widetilde{\lambda}_0) \in K \times \Lambda$ be a solution of Problem 2, unique in their first arguments, $u_\epsilon \in K$ and $u_0 \in K$, respectively. Due to (2.2) and (2.3), passing to a subsequence, we deduce that there exists $\widetilde{u} \in K$ such that $u_{\epsilon'} \rightharpoonup \widetilde{u}$ and there exists $\widetilde{\lambda} \in \Lambda$ such that $\lambda_{\epsilon'} \rightharpoonup \widetilde{\lambda}$, as $\epsilon' \to 0$.

Let $\epsilon' > 0$. We take $v = u_{\epsilon'}$ in (2.11) and $v = u_0$ in (1.1) to obtain,

$$a(u_0 - u_{\epsilon'}, u_{\epsilon'} - u_0) + b(u_{\epsilon'} - u_0, \widetilde{\lambda}_0 - \lambda_{\epsilon'}) \geq 0. \tag{2.13}$$

Recall that $(u_0, \widetilde{\lambda}_0)$ verifies (2.11)–(2.12), because, in this proof, as mentioned in the beginning, $(u_0, \widetilde{\lambda}_0)$ denotes a solution of Problem 2. Setting now $\mu = \lambda_{\epsilon'}$ in (2.12) and $\mu = \widetilde{\lambda}_0$ in (1.2), we get

$$b(u_{\epsilon'} - u_0, \widetilde{\lambda}_0 - \lambda_{\epsilon'}) \leq \epsilon' \, \|\widetilde{\lambda}_0\|_Y^2 - \epsilon' \, \|\lambda_{\epsilon'}\|_Y^2. \tag{2.14}$$

Combining (2.13)–(2.14) and taking into account the X-ellipticity of the form a, we have

$$\|u_{\epsilon'} - u_0\|_X^2 \leq \frac{\epsilon'}{m_a} (\|\widetilde{\lambda}_0\|_Y^2 - \|\lambda'_\epsilon\|_Y^2).$$

By passing to the limit as $\epsilon' \to 0$ in the relation above, we obtain that $u_{\epsilon'} \to u_0$. Due to the uniqueness of the limit, we conclude that $\widetilde{u} = u_0$.

Passing now to the limit as $\epsilon' \to 0$ in Problem 1 and keeping in mind (h_8), we deduce that $(u_0, \widetilde{\lambda}) \in K \times \Lambda$ verifies Problem 2. We conclude the proof of the theorem by considering $\lambda_0 = \widetilde{\lambda}$. □

3 The differentiable case

In this section we pay attention to the case when ϕ is a differentiable functional. Precisely, in addition to (h_1)–(h_8), we admit the following new hypothesis.

(h_9) The functional $\phi : X \to \mathbb{R}_+$ is Gâteaux differentiable, its Gâteaux gradient $\nabla\phi$ being Lipschitz continuous (of rank $L_{\nabla\phi} > 0$).

Problem 1 becomes equivalent to the following problem.

Problem 3. Given $f, h \in X$ and $\epsilon \geq 0$, find $u_\epsilon \in K \subseteq X$ and $\lambda_\epsilon \in \Lambda \subseteq Y$ such that, for all $v \in K$ and $\mu \in \Lambda$, we have:

$$\begin{aligned} a(u_\epsilon, v) + (\nabla\phi(u_\epsilon), v)_X + b(v, \lambda_\epsilon) &= (f, v)_X, \\ b(u_\epsilon, \mu - \lambda_\epsilon) - \epsilon(\|\mu\|_Y^2 - \|\lambda_\epsilon\|_Y^2) &\leq b(h, \mu - \lambda_\epsilon). \end{aligned} \tag{3.1}$$

Theorem 5. *Assume that the hypotheses (h_1)–(h_7) and (h_9) hold true. If, in addition, $\Lambda \subseteq Y$ is a bounded subset, then Problem 1 has an unique solution. Moreover,*

$$\|u_\epsilon\|_X \leq k_2, \quad \|\lambda_\epsilon\|_Y \quad \leq \frac{1}{\alpha} (\|f\|_X + M_a \, k_2 + L_{\nabla\,\phi} \, k_2 + \|\nabla\phi(0_X)\|_X), \tag{3.2}$$

where

$$k_2 = \frac{2}{m_a} \sqrt{\|f\|_X^2 + L_\phi^2 + \frac{m_a^2}{4\,r} \left(\|f\|_X^2 + \|\nabla\,\phi(0_X)\|_X^2\right) + \frac{4\,M_b^2\,r}{\alpha^2} \, \|h\|_X^2},$$

with $r = M_a^2 + L_{\nabla\,\phi}^2$.

Proof. Theorem 2 ensures the existence of a solution $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ of Problem 1 as well as the uniqueness of its first component, $u_\epsilon \in K$. We proceed by proving the uniqueness in the second component, $\lambda_\epsilon \in \Lambda$.

Let $(u_\epsilon^1, \lambda_\epsilon^1), (u_\epsilon^2, \lambda_\epsilon^2) \in K \times \Lambda$ be two solutions of Problem 1 and hence of Problem 3. For all $v \in K$ and $i \in \{1, 2\}$, we can write

$$a(u_\epsilon^i, v) + (\nabla\phi(u_\epsilon^i), v)_X + b(v, \lambda_\epsilon^i) = (f, v)_X.$$

Therefore,

$$b(v, \lambda_\epsilon^1 - \lambda_\epsilon^2) = -a(u_\epsilon^1 - u_\epsilon^2, v) - (\nabla\phi(u_\epsilon^1) - \nabla\phi(u_\epsilon^2), v)_X \quad \text{for all } v \in K.$$

According to the inf-sup property of the form b, we obtain

$$\alpha \, \|\lambda_\epsilon^1 - \lambda_\epsilon^2\|_Y \le (M_a + L_{\nabla\,\phi}) \, \|u_\epsilon^1 - u_\epsilon^2\|_X.$$

Since $u_\epsilon^1 = u_\epsilon^2$, it results that $\lambda_\epsilon^1 = \lambda_\epsilon^2$.

Let us prove (3.2). Firstly, we observe that

$$m_a \|u_\epsilon\|_X^2 \le \frac{1}{2\,p_4}\,\|f\|_X^2 + \frac{p_4}{2}\|u_\epsilon\|_X^2 + \frac{M_b^2}{2\,p_5}\|h\|_X^2 + \frac{p_5}{2}\|\lambda_\epsilon\|_Y^2 + \frac{L_\phi^2}{2\,p_6} + \frac{p_6}{2}\|u_\epsilon\|_X^2, \tag{3.3}$$

with p_4, p_5, $p_6 > 0$. Next, by (3.1) and (h_7), we get

$$\|\lambda_\epsilon\|_Y \le \frac{1}{\alpha}(\|f\|_X + M_a\,\|u_\epsilon\|_X + L_{\nabla\phi}\,\|u_\epsilon\|_X + \|\nabla\phi(0_X)\|_X). \tag{3.4}$$

Thus,

$$\|\lambda_\epsilon\|_Y^2 \le \frac{4}{\alpha^2}(\|f\|_X^2 + M_a^2\,\|u_\epsilon\|_X^2 + L_{\nabla\phi}^2\,\|u_\epsilon\|_X^2 + \|\nabla\phi(0_X)\|_X^2).$$

Setting $p_4 = p_6 = \frac{m_a}{2}$ and $p_5 = \frac{m_a\,\alpha^2}{8(M_a^2 + L_{\nabla\,\phi}^2)}$ in (3.3), we obtain

$$\frac{m_a}{4}\,\|u_\epsilon\|_X^2 \le \frac{1}{m_a}\,(\|f\|_X^2 + L_\phi^2) + \frac{4\,M_b^2(M_a^2 + L_{\nabla\,\phi}^2)}{m_a\,\alpha^2}\,\|h\|_X^2 + \frac{m_a}{4\,(M_a^2 + L_{\nabla\,\phi}^2)}\,(\|f\|_X^2 + \|\nabla\,\phi(0_X)\|_X^2).$$

Due to this last relation and (3.4), we obtain (3.2). □

Theorem 6. *Under the hypotheses* (h_1)*–*(h_7) *and* (h_9)*, if* $K \subseteq X$ *is a linear subspace, then Problem 1 has an unique solution. Moreover, (3.2) hold true.*

Proof. The existence of at least one solution is ensured by Theorem 3. For the uniqueness part we use similar arguments with those used in the proof of Theorems 2 and 5, respectively. Furthermore, as in Theorem 5, (3.2) take place. □

If $\phi \equiv 0$, then Problem 1 drives us to,

$$a(u_\epsilon, v - u_\epsilon) + b(v - u_\epsilon, \lambda_\epsilon) \ \ge\ (f, v - u_\epsilon)_X \quad \text{for all } v \in K, \tag{3.5}$$

$$b(u_\epsilon, \mu - \lambda_\epsilon) - \epsilon\|\mu\|_Y^2 + \epsilon\|\lambda_\epsilon\|_Y^2 \ \le\ b(h, \mu - \lambda_\epsilon) \quad \text{for all } \mu \in \Lambda. \tag{3.6}$$

Corollary 1. The hypotheses (h_1)–(h_3), (h_5)–(h_7) hold true. If $K \subseteq X$ is a linear subspace, then the problem (3.5)–(3.6) has an unique solution $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$.

Remark 2. Under the hypotheses (h_1)–(h_3), (h_5) and (h_6), if $\Lambda \subseteq Y$ is a bounded subset, then the problem (3.5)–(3.6) has a solution $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$, unique in its first argument.

If $\phi \equiv 0$ and $\epsilon = 0$ in Problem 1, we are lead to,

$$\begin{aligned} a(u_0, v - u_0) + b(v - u_0, \lambda_0) &\geq (f, v - u_0)_X \quad \text{for all } v \in K, && (3.7)\\ b(u_0, \mu - \lambda_0) &\leq b(h, \mu - \lambda_0) \quad \text{for all } \mu \in \Lambda. && (3.8) \end{aligned}$$

Remark 3. Under the hypotheses (h_1)–(h_3), (h_5) and (h_6), if $\Lambda \subseteq Y$ is a bounded subset, then the problem (3.7)–(3.8) has a solution $(u_0, \lambda_0) \in K \times \Lambda$, unique in its first argument.

If $K = X$ or K is a linear subspace of X, then the problem (3.7)–(3.8) drives us to

$$\begin{aligned} a(u_0, v) + b(v, \lambda_0) &= (f, v)_X \quad \text{for all } v \in K, && (3.9)\\ b(u_0, \mu - \lambda_0) &\leq b(h, \mu - \lambda_0) \quad \text{for all } \mu \in \Lambda \subseteq Y. && (3.10) \end{aligned}$$

Corollary 2. The hypotheses (h_1)–(h_3), (h_5)–(h_7) hold true. If $K = X$ or K is a linear subspace of X, then the problem (3.9)–(3.10) has an unique solution $(u_0, \lambda_0) \in X \times \Lambda$.

If $\epsilon = 0$, Problem 3 leads to the following simplified problem.

$$\begin{aligned} a(u_0, v) + (\nabla\phi(u_0), v)_X + b(v, \lambda_0) &= (f, v)_X \quad \text{for all } v \in K, && (3.11)\\ b(u_0, \mu - \lambda_0) &\leq b(h, \mu - \lambda_0) \quad \text{for all } \mu \in \Lambda. && (3.12) \end{aligned}$$

Next, we focus on the first convergence result of this section.

Theorem 7. *The hypotheses (h_1)–(h_9) hold true and $\Lambda \subseteq Y$ is a bounded subset. Let $\epsilon > 0$ and let $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ be the unique solution of Problem 1. Then, $u_\epsilon \to u_0$ and $\lambda_\epsilon \to \lambda_0$, as $\epsilon \to 0$, where $(u_0, \lambda_0) \in K \times \Lambda$ is the unique solution of the problem (3.11)–(3.12).*

Proof. As in Theorem 4, we obtain $u_\epsilon \to u_0$ as $\epsilon \to 0$. Subtracting now (3.11) from (3.1), we get

$$a(u_\epsilon - u_0, v) + (\nabla\,\phi(u_\epsilon) - \nabla\,\phi(u_0), v)_X + b(v, \lambda_\epsilon - \lambda_0) = 0.$$

Using the inf-sup property of the form b, we obtain

$$\|\lambda_\epsilon - \lambda_0\|_Y \leq \frac{M_a + L_{\nabla\,\phi}}{\alpha}\,\|u_\epsilon - u_0\|_X.$$

Therefore, $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$. □

With similar arguments we can prove the following convergence result.

Theorem 8. *The hypotheses (h_1)–(h_9) hold true and $K \subseteq X$ is a linear subspace. Let $\epsilon > 0$ and let $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ be the unique solution of Problem 1. Then, $u_\epsilon \to u_0$ and $\lambda_\epsilon \to \lambda_0$, as $\epsilon \to 0$, where $(u_0, \lambda_0) \in K \times \Lambda$ is the unique solution of the problem (3.11)–(3.12).*

4 Examples

In this section we illustrate the previous abstract results by two examples.

Let us consider $X = \{v \in H^1(\Omega),\ \gamma v = 0$ a.e. on $\Gamma_I\}$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ partitioned in three parts $\Gamma_I, \Gamma_{II}, \Gamma_{III}$ with $meas(\Gamma_i) > 0$, $i \in \{I, II, III\}$. Herein $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is the Sobolev's trace operator. We introduce also the Hilbert space,

$$S = \{\widetilde{v} = \gamma v \text{ a.e. on } \Gamma,\ \ v \in X\}.$$

Let $Y = S'$ be the dual of the Hilbert space S. The space Y is a Hilbert space too. Thus, (h_1) is fulfilled.

For Lebesgue and Sobolev spaces we use standard notation; the reader can consult, e.g., [1, 4, 16]. We also sent the reader to, e.g., [4, 6, 8] for details on Hilbert spaces. Let us introduce the bilinear forms

$$a : X \times X \to \mathbb{R},\quad a(u,v) = \int_\Omega \xi(\boldsymbol{x})\,\nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\,dx, \tag{4.1}$$

where $\xi(\boldsymbol{x}) \geq \xi^* > 0,\ \xi \in L^\infty(\Omega)$, and

$$b : X \times Y \to \mathbb{R},\quad b(v,\mu) = \langle \mu, \gamma v\rangle. \tag{4.2}$$

By $\cdot$ we denote the inner product on $\mathbb{R}^2$ and by $\langle\cdot,\cdot\rangle$ we denote the duality product between Y and S.

The form a in (4.1) verifies (h_2) with $M_a = \|\xi\|_{L^\infty(\Omega)}$ and $m_a = \xi^*$. By using the trace theorem and an inequality of Poincaré type we deduce that (h_3) holds true. The hypothesis (h_7) holds true due to the properties of the trace operator and its right invers. Moreover, keeping in mind (4.2) we immediately verify (h_8). Let $K = X$ and let Λ be the nonempty, closed, convex set of Lagrange multipliers,

$$\Lambda = \{\mu \in Y :\ \langle \mu, \gamma v\rangle \leq 0 \ \text{ for all } \ v \in \mathcal{K}_1\},$$

where $\mathcal{K}_1 = \{v \in X,\ \gamma v \leq 0$ a.e. on $\Gamma_{II}\}$. Clearly, (h_5) and (h_6) are fulfilled.

Let us define the convex, non-differential functional ϕ by means of the following relation,

$$\phi : X \to \mathbb{R}_+,\quad \phi(v) = \int_{\Gamma_{III}} |\gamma v(\boldsymbol{x})|\,d\Gamma. \tag{4.3}$$

Obviously, ϕ defined by (4.3) is a Lipschitz continuous functional. So, (h_4) is fulfilled.

Thus, all hypotheses (h_1)–(h_8) are fulfilled and this first example illustrates the abstract results in Section 2.

To proceed, we highlight the abstract results presented in Section 3 by considering a "regularized version" of the previous example. Precisely, we keep the spaces X, Y, the sets K and Λ and the definitions (4.1), (4.2), but instead of (4.3), we consider now the functional $\phi_\rho : X \to \mathbb{R}_+$ defined as follows,

$$\phi_\rho(v) = \int_{\Gamma_{III}} \left(\sqrt{(\gamma v(\boldsymbol{x}))^2 + \rho^2} - \rho\right) d\Gamma, \tag{4.4}$$

where $\rho > 0$. The functional ϕ_ρ defined in (4.4) is a Gâteaux differentiable functional for each $u \in X$. Indeed, by a standard calculus we obtain

$$\lim_{t\to 0}\frac{\phi_\rho(u+tv)-\phi_\rho(u)}{t} = \int_{\Gamma_{III}} \frac{\gamma u \gamma v}{\sqrt{(\gamma u)^2+\rho^2}}\, d\Gamma \quad \text{for all } v \in X.$$

Moreover, since

$$X \ni v \to \int_{\Gamma_{III}} \frac{\gamma u \gamma v}{\sqrt{(\gamma u)^2+\rho^2}}\, d\Gamma$$

is a linear and continuous map, then by applying the Riesz representation Theorem we conclude that there exists $\nabla\phi_\rho(u) \in X$ such that

$$\lim_{t\to 0}\frac{\phi_\rho(u+tv)-\phi_\rho(u)}{t} = (\nabla\phi_\rho(u), v)_X \quad \text{for all } v \in X.$$

According to, e.g., Proposition 1.32 in [17], in order to prove the convexity of the functional ϕ_ρ it can be proved that $\nabla\phi_\rho$ is a monotone functional, i.e., $(\nabla\phi_\rho(u) - \nabla\phi_\rho(v), u - v)_X \geq 0$ for all $u, v \in X$. Indeed, let $u, v \in X$. It is obvious that

$$\gamma u(\boldsymbol{x})\, \gamma v(\boldsymbol{x}) + \rho^2 \leq \sqrt{\gamma\, u(\boldsymbol{x})^2+\rho^2}\sqrt{\gamma\, v(\boldsymbol{x})^2+\rho^2} \quad \text{a.e. } \boldsymbol{x} \in \Gamma_{III}.$$

Then, a.e. $\boldsymbol{x} \in \Gamma_{III}$,

$$\gamma u(\boldsymbol{x})\, \gamma v(\boldsymbol{x}) - \gamma u(\boldsymbol{x})^2 \leq \sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2} - (\gamma u(\boldsymbol{x})^2+\rho^2),$$
$$\gamma u(\boldsymbol{x})\, \gamma v(\boldsymbol{x}) - \gamma v(\boldsymbol{x})^2 \leq \sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2} - (\gamma v(\boldsymbol{x})^2+\rho^2).$$

And from these last two inequalities, we are led to

$$\frac{\gamma u(\boldsymbol{x})(\gamma v(\boldsymbol{x}) - \gamma u(\boldsymbol{x}))}{\sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}} - \frac{\gamma v(\boldsymbol{x})(\gamma v(\boldsymbol{x}) - \gamma u(\boldsymbol{x}))}{\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2}} \leq 0.$$

Hence,

$$\int_{\Gamma_{III}} \left[\frac{\gamma u(\boldsymbol{x})(\gamma u(\boldsymbol{x}) - \gamma v(\boldsymbol{x}))}{\sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}} - \frac{\gamma v(\boldsymbol{x})(\gamma u(\boldsymbol{x}) - \gamma v(\boldsymbol{x}))}{\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2}}\right] d\Gamma \geq 0.$$

Therefore, $\nabla\phi_\rho$ is a monotone functional and so, ϕ_ρ is a convex functional. Moreover, since

$$\begin{aligned}|\phi_\rho(v) - \phi_\rho(w)| &= \Big|\int_{\Gamma_{III}} (\sqrt{(\gamma v)^2+\rho^2} - \sqrt{(\gamma w)^2+\rho^2})\, d\Gamma\Big| \\ &= \Big|\int_{\Gamma_{III}} \frac{(\gamma v + \gamma w)}{\sqrt{(\gamma v)^2+\rho^2} + \sqrt{(\gamma w)^2+\rho^2}}(\gamma v - \gamma w)\, d\Gamma\Big|,\end{aligned}$$

we easily deduce that ϕ_ρ is a Lispchitz continuous functional. It remains to prove that $\nabla\phi_\rho$ is a Lipschitz continuous functional. Indeed,

$$\|\nabla\phi_\rho(u) - \nabla\phi_\rho(v)\|_X = \sup_{w\in X, w\neq O_X} \frac{(\nabla\phi_\rho(u) - \nabla\phi_\rho(v), w)_X}{\|w\|_X}, \tag{4.5}$$

and

$$|(\nabla\phi_\rho(u) - \nabla\phi_\rho(v), w)_X| \le \int_{\Gamma_{III}} \left| \frac{\gamma u}{\sqrt{(\gamma u)^2+\rho^2}} - \frac{\gamma v}{\sqrt{(\gamma v)^2+\rho^2}} \right| |\gamma w|\, d\Gamma.$$

We observe that a.e. on Γ_{III},

$$\begin{aligned} &\left| \frac{\gamma u(\boldsymbol{x})}{\sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}} - \frac{\gamma v(\boldsymbol{x})}{\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2}} \right| \\ &\qquad \le \left| \frac{\gamma u(\boldsymbol{x}) - \gamma v(\boldsymbol{x})}{\sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}} \right| + \left| \frac{\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2} - \sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}}{\sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}\,\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2}}\, \gamma v(\boldsymbol{x}) \right|. \end{aligned}$$

Since $\frac{1}{\sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}} \le \frac{1}{\rho}$, $\frac{\gamma v(\boldsymbol{x})}{\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2}} \le 1$ and $\frac{\gamma v(\boldsymbol{x})+\gamma u(\boldsymbol{x})}{\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2}+\sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}} \le 1$, we obtain

$$\left| \frac{\gamma u(\boldsymbol{x})}{\sqrt{\gamma u(\boldsymbol{x})^2+\rho^2}} - \frac{\gamma v(\boldsymbol{x})}{\sqrt{\gamma v(\boldsymbol{x})^2+\rho^2}} \right| \le \frac{2}{\rho} |\gamma u(\boldsymbol{x}) - \gamma v(\boldsymbol{x})|.$$

Therefore,

$$|(\nabla\phi_\rho(u) - \nabla\phi_\rho(v), w)_X| \le \frac{2}{\rho} \int_{\Gamma_{III}} |\gamma u - \gamma v|\, |\gamma w|\, d\Gamma.$$

And from this, keeping in mind (4.5) we deduce that $\nabla\phi_\rho$ is a Lipschitz continuous functional. We conclude that all hypotheses (h_1)–(h_9) are fulfilled and the second example illustrates the abstract results in Section 3.

To end, it is worth to mention that the previous examples were "inspired" from contact models. Due to the interest into the study of the interaction between bodies, recently, several papers were devoted to the mathematical analysis of the contact models; see, for instance, [19, 20] and the references therein.

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References

[1] R.A. Adams. *Sobolev spaces.* Academic Press, 1975.

[2] Y. Bai, S. Migorski and S. Zeng. Well-posedness of a class of generalized mixed hemivariational-variational inequalities. *Nonlinear Analysis: Real World Applications*, **48**:424–444, 2019. https://doi.org/10.1016/j.nonrwa.2019.02.001.

[3] D. Braess. *Finite elements. Theory, solvers, and applications in solid mechanics.* Cambridge University Press, 2007. https://doi.org/10.1017/CBO9780511618635.

[4] H. Brézis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, 2010. https://doi.org/10.1007/978-0-387-70914-7.

[5] R. Ciurcea and A. Matei. Solvability of a mixed variational problem. *Ann. Univ. Craiova*, **36**(1):105–111, 2009.

[6] D. Cohen. *An Introduction to Hilbert Space and Quantum Logic*. Springer-Verlag New York, 1989. https://doi.org/10.1007/978-1-4613-8841-8.

[7] M. Chivu Cojocaru and A. Matei. Well-posedness for a class of frictional contact models via mixed variational formulations. *Nonlinear Analysis: Real World Applications*, **47**:127–141, 2019. https://doi.org/10.1016/j.nonrwa.2018.10.009.

[8] L. Debnath and P. Mikusiński. *Introduction to Hilbert spaces with Applications-3rd Edition*. Elsevier Academic Press, 2005.

[9] I. Ekeland and R. Témam. *Convex Analysis and Variational Problems*. Classics in Applied Mathematics, SIAM, 28, 1999. https://doi.org/10.1137/1.9781611971088.

[10] J. Haslinger, I. Hlaváček and J. Nečas. *Numerical methods for unilateral problems in solid mechanics, in: P.G. Ciarlet, J.-L. Lions (Eds.), Handbook of Numerical Analysis, Vol. IV, 313–485*. North-Holland, Amsterdam, 1996. https://doi.org/10.1016/S1570-8659(96)80005-6.

[11] A. Matei. Weak solvability via Lagrange multipliers for contact problems involving multi-contact zones. *Mathematics and Mechanics of Solids*, **21**(7):826–841, 2016. https://doi.org/10.1177/1081286514541577.

[12] A. Matei. A mixed hemivariational-variational problem and applications. *Computers and Mathematics with Applications*, **77**(11):2989–3000, 2019. https://doi.org/10.1016/j.camwa.2018.08.068.

[13] A. Matei, S. Sitzmann, K. Willner and B. Wohlmuth. A mixed variational formulation for a class of contact problems in viscoelasticity. *Applicable Analysis*, **97**(8):1340–1356, 2018. https://doi.org/10.1080/00036811.2017.1359569.

[14] A. Matei and M. Sofonea. A mixed variational formulation for a piezoelectric frictional contact problem. *IMA Journal of Applied Mathematics*, **82**(2):334–354, 2017.

[15] S. Migorski, Y. Bai and S. Zeng. A class of generalized mixed variational-hemivariational inequalities II: Applications. *Nonlinear Analysis: Real World Applications*, **50**:633–650, 2019. https://doi.org/10.1016/j.nonrwa.2019.06.006.

[16] S. Migorski, A. Ochal and M. Sofonea. *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*. Springer, 2013. https://doi.org/10.1007/978-1-4614-4232-5.

[17] M. Sofonea and A. Matei. *Mathematical Models in Contact Mechanics*. Cambridge University Press, 2012.

[18] M. Sofonea, A. Matei and Y. Xiao. Optimal control for a class of mixed variational problems. *Zeitschrift für angewandte Mathematik und Physik*, **70**(4):127, 2019. https://doi.org/10.1007/s00033-019-1173-4.

[19] M. Sofonea, Y.B. Xiao and M. Couderc. Optimization problems for a viscoelastic frictional contact problem with unilateral constraints. *Nonlinear Analysis: Real World Applications*, **50**:86–103, 2019. https://doi.org/10.1016/j.nonrwa.2019.04.005.

[20] Y.B. Xiao and M. Sofonea. Generalized penalty method for elliptic variational hemivariational inequalities. *Appl Math Optim*, 2019. https://doi.org/10.1007/s00245-019-09563-4.