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1. Introduction

Many of structures being used in various fields of engineering are composed of different beam-type structural members. Analysis and design of these structures involves basic statements and relations of beam theory. On the other hand, beam theory may be used for the modelling of complex structures containing large amount of different members by simple beam-type models. It is necessary to remark, that classical beam theories may be applied for the structural members, the shape and geometric relations of which lie within the certain boundaries. On the limits of these restrictions, the validity of certain assumptions requires careful examination. Therefore, much effort has been made to the development of alternative beam theories and corresponding finite elements.

The development of finite elements based on the semi-analytical theory is considered here. The semi-analytical finite elements and the corresponding higher-order beam theory are suggested and already presented in [1-3]. This theory describes the cross-sectional distribution of three-dimensional variables by the use of semi-analytical finite element approximations. To distinguish it from the conventional beam finite elements which are based on the engineering theories, a new term, *semi-analytically based finite elements* (SABFE), is introduced. Development of SABFE may be considered as further step in application of semi-analytical theory. The functional (semi-discrete) beam equations and general discrete models as well as their applications to thin-walled beams are presented in this paper.

2. Functional (Semi-Discrete) Equations

The functional (semi-discrete) mathematical model of the beam is a typical many-field problem. The mixed formulation [4] is the most suitable tool for deriving the governing equations of the coupled problem which is expressed in terms of both primary and secondary variables. The two-level hierarchical approach is proposed in order to develop a semi-discrete model. All the matrices and the vectors referred to here to define the primary variables will be denoted by the subscript 1. The longitudinal three-dimensional displacement $u_x(x, y, z)$, the normal stress $\sigma_x(x, y, z)$ and strains $\varepsilon_x(x, y, z)$ are assumed to be as functional primary variables. The remaining displacements and stresses may be hierarchically involved in the problem and will be referred to as secondary variables. The semi-discrete analogous of them will be denoted by the subscript 2.

After independent approximation of displacement and stress fields the primary state variables are described by the corresponding semi-discrete variables such as the generalised displacements $\boldsymbol{U}_1(x)$, the generalised strains $\Theta_1(x)$ and the generalised stresses $\boldsymbol{Q}_1(x)$. Taking into account the symmetry of primary differential compatibility operator $[\boldsymbol{B}_{cM}]^t$ and equilibrium operator $[\boldsymbol{B}_{eM}]$, thus $[\boldsymbol{B}_{eM}] \equiv [\boldsymbol{B}_{cM}]^t \equiv [\boldsymbol{B}_1]$, the set of governing functional (semi-discrete) primary equations contain equilibrium (static), compatibility (kinematic) as well as constitutive (physical) relationships with corresponding boundary conditions and are written as follows ([1-3]) :

equilibrium equations

$$[\boldsymbol{B}_1(x)]\boldsymbol{Q}_1(x) = -\boldsymbol{p}_1(x); \tag{1a}$$

compatibility equations

$$[\boldsymbol{B}_1(x)]\boldsymbol{U}_1(x) - \Theta_1(x) = \boldsymbol{O}; \tag{1b}$$

constitutive equations

$$[\boldsymbol{D}_1(x)]\boldsymbol{Q}_1(x) - \Theta_1(x) = \boldsymbol{O}. \tag{1c}$$

Here $[\boldsymbol{D}]$ and $\boldsymbol{p}$ are flexibility matrix and external load vectors. The distribution of three-dimensional variables may be recovered from the semi-discrete state variables $\boldsymbol{U}_1(x)$ and $\boldsymbol{Q}_1(x)$.

The secondary fields are described by the corresponding semi-discrete variables such as the displacements $\boldsymbol{U}_2(x)$, the generalised strains $\Theta_2(x)$ and the generalised stresses $\boldsymbol{Q}_2(x)$. After introducing the assumption about an independent constitutive relationship between the primary and the secondary variables (diagonal elasticity tensor), which applies to the existing beam theories, the coupling between the primary and secondary variables is determined by an algebraic operator $[\boldsymbol{B}_{12}]$ appearing in the compatibility and the equilibrium equations. With the above remarks in mind, the secondary problem may be defined by the following equations:

equilibrium equations

$$\begin{bmatrix} [\boldsymbol{B}_1(x)] & [\boldsymbol{B}_{12}(x)] \\ [\boldsymbol{O}] & [\boldsymbol{B}_{22}(x)] \end{bmatrix} \begin{Bmatrix} \boldsymbol{Q}_1(x) \\ \boldsymbol{Q}_2(x) \end{Bmatrix} = -\begin{Bmatrix} \boldsymbol{p}_1(x) \\ \boldsymbol{p}_2(x) \end{Bmatrix}; \tag{2a}$$

compatibility equations

$$\begin{bmatrix} [\boldsymbol{B}_1(x)] & [\boldsymbol{O}] \\ -[\boldsymbol{B}_{12}(x)]^t & [\boldsymbol{B}_{22}(x)] \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_1(x) \\ \boldsymbol{U}_2(x) \end{Bmatrix} - \begin{Bmatrix} \Theta_1(x) \\ \Theta_2(x) \end{Bmatrix} = \boldsymbol{O}; \tag{2b}$$

constitutive equations

$$\begin{bmatrix} [D_1(x)] & [O] \\ [O] & [D_2(x)] \end{bmatrix} \begin{Bmatrix} Q_1(x) \\ Q_2(x) \end{Bmatrix} - \begin{Bmatrix} \Theta_1(x) \\ \Theta_2(x) \end{Bmatrix} = O. \tag{2c}$$

The secondary model (2) contains the uncoupled primary compatibility (1b) as well as the constitutive (1c) relationships that may be applied for the elimination of the variables $Q_1(x)$ and $\Theta_1(x)$. When some of the mathematical operations have been performed, the first coupled secondary equilibrium equation may be expressed in terms of primary displacements $U_1(x)$. The secondary strains $\Theta_2(x)$ are eliminated using relation (2c). Finally, the secondary model is expressed as the three-field problem

$$\begin{bmatrix} [B_1(x)][D_1(x)]^{-1}[B_1(x)] & [O] & [B_{12}(x)] \\ [O] & [O] & [B_{22}(x)] \\ -[B_{12}(x)]^{t} & [B_{22}(x)] & -[D_2(x)] \end{bmatrix} \begin{Bmatrix} U_1(x) \\ U_2(x) \\ Q_2(x) \end{Bmatrix} = - \begin{Bmatrix} p_1(x) \\ p_2(x) \\ O \end{Bmatrix}. \tag{3}$$

This equation will be used for the development of mixed SABFE for thin walled beams.

3. General Finite Element Relations

A number of distinct procedures is available for obtaining of the discrete finite element model associated with the equation (3). By introducing Lagrangian multipliers, we shall deal with a constraint variational principle. This obvious technique leads to the creation of variational principle for any set of linear equations. By treating equation (3) as a set of constraints, we can establish a general variational principle simply by putting the initial functional $\Pi = 0$. The Lagrangian multipliers corresponding to the first equation (3) are easily identified as primary displacements $\lambda_1(x) \equiv U_1(x)$, multipliers corresponding to the second equation as secondary displacements $\lambda_2(x) \equiv U_2(x)$ and multipliers corresponding to the third equation as secondary generalised stresses $\lambda_3(x) \equiv Q_2(x)$. Now the constraint variational principle may be expressed by the following functional

$$\begin{aligned} \tilde{\Pi}(U_1(x), U_2(x), Q_2(x)) &= \\ &= \int_L (U_1(x))^{t}\Big([B_1(x)][D_1(x)]^{-1}[B_1(x)]U_1(x) + [B_{12}(x)]U_2(x) + p_1(x)\Big)\mathrm{d}L + \\ &+ \int_L (U_2(x))^{t}\Big([B_{22}(x)]^{t}Q_2(x) + p_2(x)\Big)\mathrm{d}L + \\ &+ \int_L (Q_2(x))^{t}\Big(-[B_{12}]^{t}(x)U_1(x) + [B_{22}(x)]U_2(x) - [D_2(x)]Q_2(x)\Big)\mathrm{d}L + \text{bound. t.} \end{aligned} \tag{4}$$

The next step in the discretisation of equation (3) is to choose the appropriate shape functions for the longitudinal distribution. In order to solve the beam problem as a mixed one, we can start directly to make the independent approximation of each variable $U_1(x)$, $U_2(x)$ and $Q_2(x)$. Practical experience and theoretical knowledge obtained from beam, plate and shell analysis may be generalised and applied here. It is possible and convenient to introduce the approximations with the shape functions of a general character

$$
\begin{aligned}
U_1(x) &= \left[N_{U1}(x)\right]U_1,\\
U_2(x) &= \left[N_{U21}(x)\right]U_1 + \left[N_{U22}(x)\right]U_2,\\
Q_2(x) &= \left[N_{Q2}(x)\right]Q_2.
\end{aligned}
\tag{5}
$$

Here $[N_{U1}(x)]$, $[N_{U21}(x)]$, $[N_{U22}(x)]$ and $[N_{Q2}(x)]$ are the shape matrices while U_1, U_2 and Q_2 are the vectors of the discrete variables.

The most complicated issue is the approximation of secondary displacements. The second expression of (5) allows to use us the higher-order interpolation polynomials and provides a link between primary and secondary variables. By the selection of the appropriate shape functions, both the conforming finite elements with C_0 as well as C_1 continuity or the non-conforming elements may be imposed by the same approximation (5).

The algebraic expression of functional (4) is achieved by the substitution of approximations (5). The variations of the functional produce the set of three simultaneous equations

$$
\begin{Bmatrix}
\dfrac{\partial\tilde{\Pi}}{\partial U_1}\\[2mm]
\dfrac{\partial\tilde{\Pi}}{\partial U_2}\\[2mm]
\dfrac{\partial\tilde{\Pi}}{\partial Q_2}
\end{Bmatrix} = O.
$$

Finally, after integration by parts

$$
\begin{bmatrix}
[A] & [O] & [H]\\
[O] & [O] & [C]^t\\
[H]^t & [C] & [G]
\end{bmatrix}
\begin{Bmatrix} U_1\\ U_2\\ Q_2 \end{Bmatrix}
=
\begin{Bmatrix} F_1\\ F_2\\ O \end{Bmatrix}.
\tag{6}
$$

Here, algebraic submatrices in the coefficient matrix are

$$
[A] = \int_L \left(\left[B_1(x)\right]\left[N_{U1}(x)\right]\right)^t \left[D_1(x)\right]^{-1}\left[B_1(x)\right]\left[N_{U1}(x)\right]\mathrm{d}L; \tag{7a}
$$

$$[H] = -\int_L [N_{U1}(x)]^t [B_{12}(x)][N_{Q2}(x)]\, \mathrm{d}L - \int_L ([B_{22}(x)][N_{U21}(x)])^t [N_{Q2}(x)]\mathrm{d}L; \quad (7b)$$

$$[C] = -\int_L [N_{Q2}]^t [B_{22}(x)][N_{U22}(x)]\mathrm{d}L; \quad (7c)$$

$$[G] = \int_L [N_{Q2}(x)]^t [D_2(x)][N_{Q2}(x)]\mathrm{d}L. \quad (7d)$$

The free terms remain as follows

$$F_1 = \int_L [N_{U1}(x)]^t p_1(x)\, \mathrm{d}L + \int_L [N_{U21}(x)]^t p_2(x)\mathrm{d}L + F_{1F}; \quad (8a)$$

$$F_2 = \int_L [N_{U22}(x)]^t) p_2(x)\mathrm{d}L - F_{2F}. \quad (8b)$$

where F_{1F} and F_{2F} denote the vectors of external nodal loads.

Expressions (7) and (8) contain general shape matrices and involve an integration over the whole domain considered. Practically, the shape functions are defined locally and this integration is changed by the summation of integrals obtained over the individual elements. Further transformations of the model (6) depend on the properties of the interpolation polynomials. If any component of the unknown variable cannot be eliminated still leaving a well-defined problem, then the mixed problem can be transformed into an irreducible formulation. The elimination of secondary stresses is the path to formulate an reducible model. The selection of the appropriate rules (some times strongly heuristic) for the elimination of the stresses and the application of different integration techniques is widely discussed in bending problems, where the shear forces have to be eliminated. In this context, the elimination of stresses at the element level by using condensation technique is often more effective than the application of special integration rules.

The general solution of coupled problem (6) consists of two ingredients

$$U_1 = U_{1(1)} + U_{1(2)},$$

where the primary contribution is obtained from the solution of the discretised uncoupled model (1) which is written as

$$[A]U_{1\,(1)} = F_1, \quad (9)$$

while the secondary contribution is

$$U_{1(2)} = -[A]^{-1}[H]\, Q_2.$$

The primary model (9) may be derived and solved independently using a standard displacement approach, for example the Lagrangian variational principle. In the absence of secondary loads ($F_2 = O$), the equilibrium equation provides zero secondary stresses

$$Q_2 = O$$

and zero secondary displacement contribution

$$U_{1(2)} = O.$$

In general, Q_2 has to be found by iterations, while in the particular statically determined case Q_2 follows directly from the second (equilibrium) equation (6). The secondary displacements U_2 have to be found from the third (compatibility) equation (6).

4. Application to Thin-Walled Beams

Previously, we have dealt theoretically with the SABFE of beams. Now we shall demonstrate how general relations can be applied to thin-walled beams. The first step in the implementation of this approach is the development of primary elements.

Let us consider a straight thin-walled beam with a constant cross-section. The beam is referred to the local Cartesian co-ordinate system, the axis Ox of which coincides with the beam axis while the local normalised co-ordinate ξ is used for the description of an individual element. In terms of the conventional finite elements, any element of a beam will be globally defined by two nodal points i and j lying on the beam axis (fig. 1a). In fact, the global node indicates an appropriate cross-section. In the most general case, the node j of the finite element contains n degrees of freedom, referred here to n subnodes (fig. 1b). The term subnode is used to identify node k of the cross-section j. The subnode is indicated by the two subscripts jk.

The vector of the nodal displacements of the element e is defined by $U_e \equiv \{U_{ei}, U_{ej}\}^t$, where $U_{ei} \equiv \{U_{ei1}, U_{ei2}, \ldots, U_{eik}, \ldots, U_{ein}\}^t$. In a particular case, the subnode vector U_{eik} is defined by the single variable, thus $U_{eik} \equiv \{U_{eik}\}$. According to [2], the functional compatibility operator of the beam $[B_1(x)] \equiv [B_1]$ includes only the first-order derivatives

$$[B_1] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial x} \end{bmatrix}. \qquad (10)$$

therefore C_o continuity requirements for displacement approximation are sufficient. All displacement components are interpolated by the same first-order polynomials, thereby the shape matrix $[\boldsymbol{N}_{U1}(x)] \equiv [\boldsymbol{N}_{U1e}(\xi)]$ is defined by n rows and $2n$ columns and has a regular pattern

$$[\boldsymbol{N}_{U1e}(\xi)] = \begin{bmatrix} N_{e1}(\xi) & & & N_{e2}(\xi) & & \\ & N_{e1}(\xi) & & & N_{e2}(\xi) & \\ & & \cdots & & & \cdots \\ & & N_{e1}(\xi) & & & N_{e2}(\xi) \end{bmatrix}. \tag{11}$$

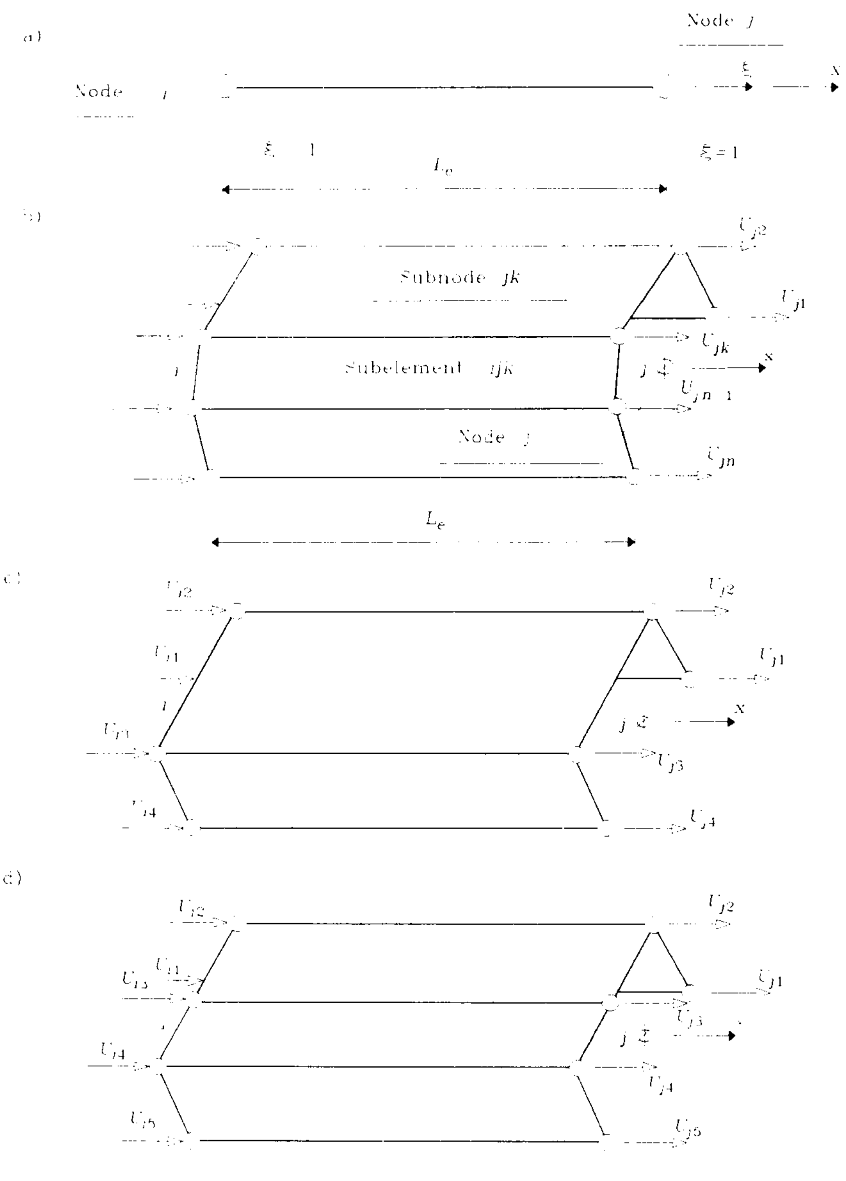

Fig. 1. Semi-analytically based primary finite elements of a thin-walled beam:

a) global model; b) general model with degrees of freedom; c) element THIN3; d) element THIN4

Here, the first-order Lagrangian interpolation polynomials may be applied as shape functions N_{ei} Their derivatives are found by following standard differentiation rules

$$\frac{\partial N_{e1}(\xi)}{\partial x} = -\frac{1}{L_e}, \quad \frac{\partial N_{e2}(\xi)}{\partial x} = \frac{1}{L_e}, \tag{12}$$

where L_e is the element length. Taking into account the diagonal pattern of the approximation matrix (11) and of the compatibility operator (10), the strain-displacement relation matrix for element e

$$\left[\boldsymbol{B}_e(\xi)\right] = \left[\boldsymbol{B}_1\right]\left[\boldsymbol{N}_{U1e}(\xi)\right]$$

is also regular. By substituting the derivatives (12), the matrix $[\boldsymbol{B}_e(\xi)] \equiv [\boldsymbol{B}_e]$ is independent of co-ordinates and expressed as follows

$$\left[\boldsymbol{B}_e\right] = \frac{1}{L_e}\left[-[\boldsymbol{I}] \quad [\boldsymbol{I}]\right]. \tag{13}$$

The stiffness matrix $[\boldsymbol{K}_e] \equiv [\boldsymbol{A}]$ of the whole primary element is expressed according to (7a). When the substitution of the matrices $[\boldsymbol{B}_e]$ established by (13) and $[\boldsymbol{D}_1]$ derived in [2] as well as the integration along the length L_e have been made, the final stiffness matrix of the element e is expressed in terms of the dimensional parameters such as Young's modulus E_e, the length L_e, the characteristic dimension of the cross-section H_e as well as the wall thickness t_e and in terms of the nondimensional coefficients

$$\left[\boldsymbol{K}_e\right] = \frac{E_e t_e H_e}{L_e}\begin{bmatrix} [\overline{\boldsymbol{C}}_e] & -[\overline{\boldsymbol{C}}_e] \\ -[\overline{\boldsymbol{C}}_e] & [\overline{\boldsymbol{C}}_e] \end{bmatrix}. \tag{14}$$

Here, $\left[\overline{\boldsymbol{C}}_e\right]$ is the matrix of nondimensional coefficients that depends on the geometry of the appropriate cross-section.

The generalised stress-displacement relationship is expressed as usual on the element level as

$$\boldsymbol{Q}_e = \left[\beta_{Qe}\right]\boldsymbol{U}_e. \tag{15}$$

In the absence if internal loads only the generalised relation matrix $[\beta_{Qe}]$ has to be established and expressed in the following form

$$\left[\beta_{Qe}\right] = \left[\boldsymbol{D}_1\right]^{-1}\left[\boldsymbol{B}_e\right].$$

By substituting the matrices $[\boldsymbol{D}_1]$ and $[\boldsymbol{B}_e]$ we finally obtain

$$\left[\beta_{Qe}\right] = \frac{E_e t_e H_e}{L_e}\left[-[\overline{\boldsymbol{C}}_e] \quad [\overline{\boldsymbol{C}}_e]\right]. \tag{16}$$

The computation of the stress vector S_e for the element e will be implemented in the same manner as in the case of vector Q_e using the similar expression as (15)

$$S_e = [\beta_{Se}]U_e. \tag{17}$$

Here, the stress-displacement relation matrix $[\beta_{Se}]$ is defined as

$$[\beta_{Se}] = \begin{bmatrix} [\bar{C}_e] & \\ & [\bar{C}_e] \end{bmatrix}^{-1} [\beta_{Qe}].$$

The substitution of particular matrices expressed in terms of the finite element parameters provides the final expression of the stress matrix

$$[\beta_{Se}] = \frac{E_e}{L_e}[-[I] \quad [I]]. \tag{18}$$

A final remark concerns the assembling of SABFE. In general, the assembling of the global matrices is a two-level procedure. The first step involves the assembling of the semi-analytical finite elements while the second step deals with the assembling of the conventional finite elements. In fact, the global model may be considered as an assemblage of individual subelements. Here, the subelement is considered as a part of SABFE defined by a single semi-analytical element. For this reason, the definition of subelements and their characteristic matrices may be sufficient for the total assembling of the global model.

The possibilities of constructing various SABFE are ample but the present work deals only with two elements, THIN3 and THIN4 (fig. 1c, d) intended for U-type cross-section. They were derived using expressions (14) to (18).

The development of the secondary finite elements is a further step of the complex numerical analysis. The higher-order equations obtained by the secondary semi-analytical finite elements have to be used for the evaluation of the model (3). As we have already pointed out, a large number of different models may be formulated to describe various types of mechanical behaviour. A thin-walled beam assembled from semi-analytical elements of the Timoshenko-type ([2]) will be considered as an illustration.

Let us define vectors and operators for a single subelement. The semi-discrete equations (2) written for a single semi-analytical subelement may be considered as an analytical model. After separation of the coupled model, the longitudinal variables are selected as primary components and described by the generalised displacements $U_1(x) \equiv \{U_{x1}(x), U_{x2}(x)\}^t$, the generalised strains $\Theta_1(x) \equiv \{\Delta_{x1}(x), \Delta_{x2}(x)\}^t$ and the generalised stresses $Q_1(x) \equiv \{N_{x1}(x), N_{x2}(x)\}^t$. The transversal variables are selected as secondary components and described by the vectors $U_2(x) \equiv \{U_y(x)\}$, $\Theta_2(x) \equiv \{\Theta_z(x)\}$ and $Q_2(x) \equiv \{Q_y(x)\}$.

The basic secondary operators in (2) are simply defined as

$$[\boldsymbol{B}_{12}(x)] = \begin{bmatrix} \dfrac{1}{h(x)} \\ -\dfrac{1}{h(x)} \end{bmatrix}; \qquad [\boldsymbol{B}_{22}] = \left[\dfrac{\partial}{\partial x}\right].$$

The choice of the appropriate shape functions follows a well-defined path. Thus, let us consider the subelement ij defined by the following nodal variables: longitudinal displacements $\boldsymbol{U}_1 \equiv \{U_{xi1}, U_{xi2}, U_{xj1}, U_{xj2}\}^t$, transversal displacements $\boldsymbol{U}_2 \equiv \{U_{yi}, U_{yj}\}^t$ and transversal force $\boldsymbol{Q}_2 \equiv \{Q_y\}$ (fig. 2). For this element the standard linear approximation (5a) used for longitudinal displacements is expressed in the local co-ordinate ξ by Lagrangian polynomials, where shape matrix $[\boldsymbol{N}_{U1}(x)]$ is defined by (11). The transversal force is obviously taken as constant, while $[\boldsymbol{N}_{Q2}(x)] \equiv [\boldsymbol{N}_{Q2}(\xi)]$ is simply

$$[\boldsymbol{N}_{Q2}(\xi)] = [1]. \tag{19}$$

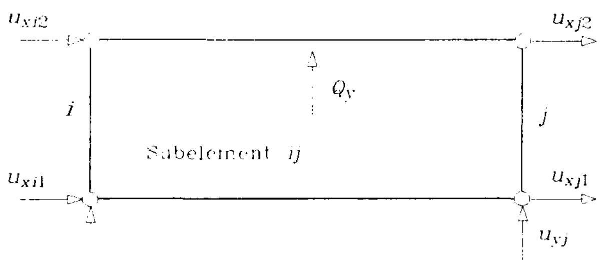

Fig. 2. The subelement of Timoshenko-type and the basic nodal variables

Originally, as stated in [5], the transversal displacements were approximated by the linked linear-quadratic polynomial expression. This kind of shape functions was initially proposed for membrane elements with drilling degrees of freedom. In our case, the angular displacements θ_{zi} and θ_{zj} have to be expressed in terms of longitudinal nodal variables, i.e.

$$\theta_{zi} = (u_{xi} - u_{xj})/h.$$

The final expressions of the corresponding shape matrices are written as follows

$$[\boldsymbol{N}_{U21}(\xi)] = \frac{1}{16}\frac{1}{h}(1-\xi^2)[1 \quad -1 \quad -1 \quad 1]; \quad [\boldsymbol{N}_{U22}(\xi)] = [N_{e1}(\xi) \quad N_{e2}(\xi)]. \tag{20}$$

The above shape matrices are used for the assembling of global matrices and vectors.

5. Numerical Examples

For a critical evaluation of the semi-analytical beam theory as well as for the demonstration of the capabilities of SABFE, the numerical tests have been performed. The previously described finite elements of beam THIN3 and THIN4 are used for the modelling of thin-walled beams with a U-type cross-section.

<u>*Example 1*: linear analysis of a cantilever beam under compression caused by the concentrated force.</u> A thin-walled cantilever beam is considered in this example (fig. 3). For the sake of comparison with an engineering theory, a Cartesian co-ordinates system is provided to the bending center of the cross-section. By applying a standard technique of dimensional analysis, the behaviour of the beam is described in terms of a nondimensional model. The following parameters have been taken as model variables: the characteristic dimension of the cross-section H, the nondimensional length of the element $\bar{L}$, the nondimensional height of the web $\bar{b}$, the nondimensional wall thickness $\bar{t}$ and the dimensional material constant - Young's modulus E. A vertical concentrated load F is applied to the free end. In the particular case considered, the values of nondimensional variables are taken as $\bar{L}$ = 8.0, $\bar{b}$ = 0.5 and $\bar{t}$ = 0.1. Due to a nondimensional origin of the model, the dimensional parameters used in the numerical analysis are also simplified H = 1, E = 1 and F = 1. Only two continuous state variables such as longitudinal displacement $u_x(x)$ and normal stress $\sigma_x(x)$ are taken into consideration. According to the equations of the classical beam theory, the solution of this problem has to be independent on the longitudinal co-ordinate x.

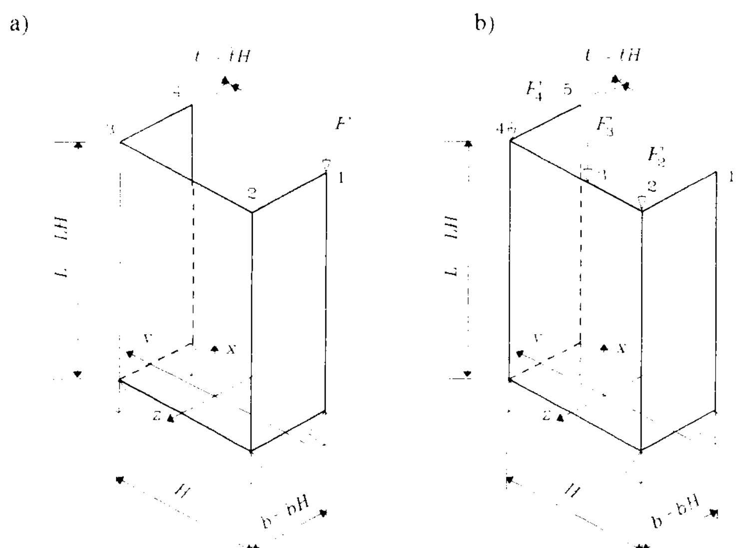

Fig. 3. Thin-walled beam with U-type cross-section:

a) illustration of example 1a with 3 SFE; b) illustration of example 1b with 4 SFE

Two loading cases are studied in this example. The eccentric load F added to the web in the point 1 is considered in the first case (fig. 3a). Different distributions of nondimensional stress $\overline{\sigma}_x(x) = \sigma_x(x)H^2 / F$ over the perimeter of the cross-section are obtained using different models (see fig. 4a). The numerical values are shown in the table 1.

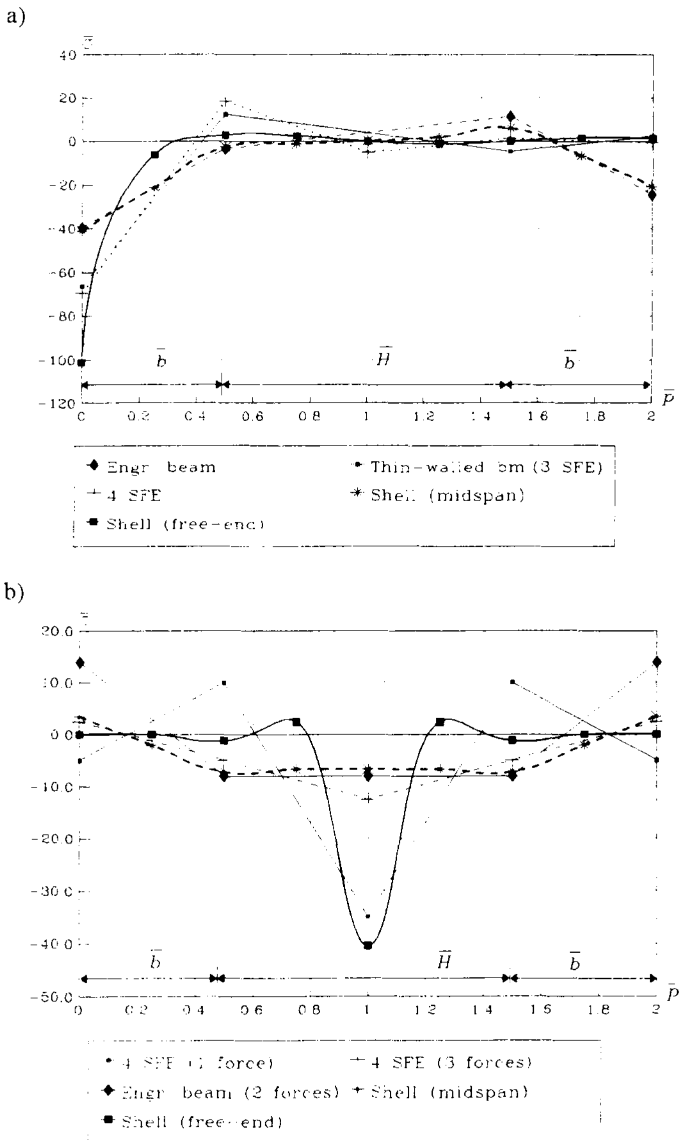

Fig. 4. Perimetric distribution of normal stresses $\overline{\sigma}_x$ in the U-type cross-section:

a) under concentrated eccentric load; b) under concentrated central load

By using the SABFE method, the displacement field is calculated by the solution of the algebraic model (9) while stresses and generalised stresses are calculated by the general matrix expression (15) and (17), in which component matrices are defined by (16) and (18). Stress distribution corresponding to the classical and thin-walled beam theories is calculated analytically, where the contributions of the individual generalised stresses such as the axial force $N = F$, two bending moments $M_y = 3FH/4$, $M_z = FH/2$ and the bimoment $B = 5FH^2/32$ are indicated separately. The subdivision of the cross-section into three SFE denoted by nodes 1-4 (fig. 3a) coincides with the theory of thin-walled beams. The subdivision of the cross-section into four SFE denoted by nodes 1-5 as in fig. 3b has no analogy in the existing beam theories and may be treated as a more exact solid model.

Table 1. Values of normal nondimensional stresses $\overline{\sigma}_x$ in a U-type cross-section for example 1 with eccentric load

Nodal points	Contribution of individual forces				Total values				
	Axial force N	Bending moment M_y	Bending moment M_z	Bimoment B	Classical theory	Thin-walled theory	SFEM (3 SFE)	Shell theory (in mid-span)	Shell theory (at free end)
1	-5.0	-27.0	-7.5	-26.78	-39.5	-66.28	-66.30	-40.97	1.33
2	-5.0	9.0	-7.5	16.07	-3.5	12.57	12.60	-2.69	0.01
3	-5.0	9.0	7.5	-16.07	11.5	-4.57	-4.57	6.07	2.97
4	-5.0	-27.0	7.5	26.78	-24.5	2.28	2.29	-20.52	-101.

Finally, the whole beam is investigated as a thin shell discretised by rectangular shell elements. The shell model is considered as an exact model of the complex three-dimensional solid problem. The spatial distribution of the exact field of normal stress σ_x is demonstrated in fig. 5. The white colour corresponds to the tensile stresses. It is of interest to note, that the longitudinal distribution in this example (fig. 5a) produces three different domains. The stresses in the midspan domain are almost exactly described by the classical engineering theory while the thin-walled theory provides the enveloping approximation of stresses in the end domains with the exception of the loading point.

The second loading case (fig. 3b) deals with the load F applied to the midpoint (point 3) of the middle of the central wall. This problem cannot be solved in the framework of existing beam theories while SABFE may successfully be applied. The nearest solution is possible by the approximation of the load F by the two concentrated forces $F_2 = 0.5F$ and $F_4 = 0.5F$. This type of loading provides zero bimoment and the identical solution of both technical and thin-walled beam theories. In a framework of SFE, two load approximations may be considered. The first one is the exact representation of the initial load expressed as $F_3 = F$, the second is the three-load representation $F_2 = 0.25F$, $F_3 = 0.5F$ and $F_4 = 0.25F$. The distribution of stresses for different models is presented in fig. 4b. The results obtained demonstrate the sensitivity of stresses due to local loading where the small imperfections of loading points may lead to significant changes of the stresses. The results deduced for a three-load discretisation agree well with the existing beam theories. The spatial distribution of stresses obtained by the shell model is also demonstrated in fig. 5b.

a) b)

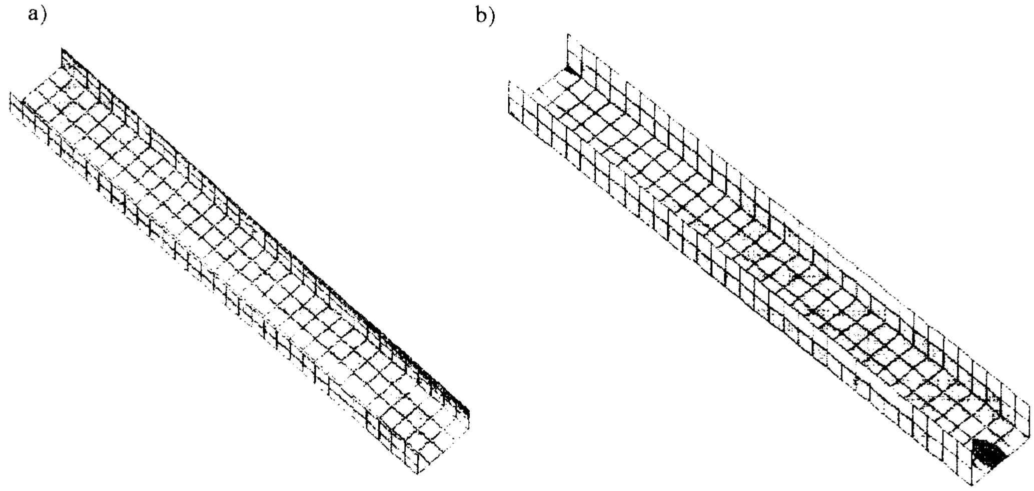

Fig. 5. Space distribution of nondimensional stresses $\overline{\sigma}_x$ obtained by shell model:
a) illustration of example 1 under concentrated eccentric load; b) illustration of example 2 under concentrated central load

Example 2 mode shape analysis of a thin-walled beam. Free vibrations of a thin-walled cantilever beam with a U-type cross-section are considered (fig. 6). The details of the geometry are described in previous examples while the basic nondimensional parameters are taken as $\overline{L}$ = 1.0, $\overline{b}$ = 0.5 and $\overline{t}$ = 0.1. A Cartesian co-ordinate system is attached to the bending centre of the cross-section O. The three-dimensional state variables and equations are formed identically and discretisation is also performed in the same manner.

Two eccentric concentrated masses m with the two longitudinal dynamic degrees of freedom are added to the points 1 and 2. The free-vibration analysis performed in primary variables supplies two vibration modes. In the framework of the engineering beam theories any mass has to be attached to the bending centre and provides three dynamic degrees of freedom. If the global mass matrix is assembled by standard algebraic summation of individual masses, some of the dynamic properties of current structure are lost and the classical theory cannot represent the true dynamic behaviour of the initial system.

In contrast, in a framework of the SFE method, the global displacements at node are defined by the four-dimensional vector of the generalised variables $\boldsymbol{U}(x) \equiv \{U_{x1}(x), U_{x2}(x), U_{x3}(x), U_{x4}(x)\}^t$. The corresponding diagonal lumped mass matrix exactly reflects mass properties of the initial system, finally yielding at global node

$$\boldsymbol{M} = \mathrm{diag}\begin{bmatrix} m & m & 0 & 0 \end{bmatrix}.$$

Thus, SFE method allows describe systems simply in a way which is different from that considered by classical beam theory.

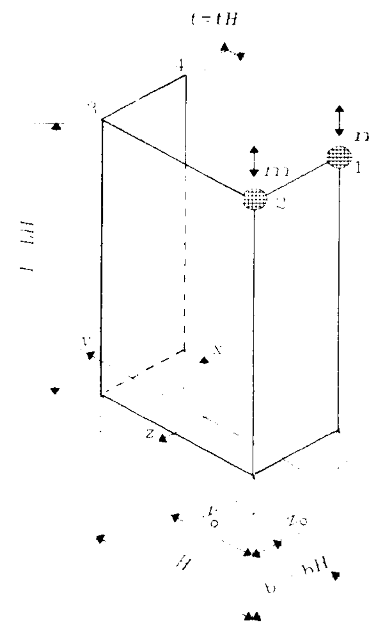

Fig. 6. Geometry of example for free-vibration problem

As already explained, this beam in general, is an example of the coupled axial-bending model where longitudinal movements lead to transversal displacements. The model describing their pure axial behaviour is initially based on the simplest primary semi-analytical elements. The beam is discretised by SABFE primary elements THIN3. The determination of the transversal displacements calls the higher-order secondary models. For this purpose, a thin-walled beam is considered an assemblage of semi-analytical elements of the Timoshenko type. A general approximation (20) is assigned to the longitudinal distribution of transversal displacements. In the absence of transversal forces, the general secondary model (6) turns into an "incompressible" model with $[G] = [O]$ while the transversal displacement components U_2 are deduced from the solution of a particular equation

$$[H]^t\,[U]_1 - [C]\,[U]_2 = [O].$$

In our case, this equation expressed for an individual subelement of the finite element ij is simple turned to the recurrent formula

$$u_{2_j} = u_{2_i} - \left[H_{ij}\right]^t \left[U_{1_{ij}}\right].$$

The final values of the secondary displacements depend on the element length. The transversal movement of free-end point 1 is illustrated in fig. 7.

In addition to bending, the coupled warping-torsion deformations occur in this example. It may be treated as a third-level effect which can also be established on the results of the second-level model by following simple geometric considerations. The rotation angle due to torsion is established from transversal displacements u_{p1} and u_{p3} of semi-analytical elements 1-2 and 3-4 respectively as

$$\varphi(x) = \left(u_{p3}(x) - u_{p1}(x)\right) \cdot h \ .$$

The correction of transversal components of the displacements due to torsion may simply be added to the initial values. This effect as illustrated in fig. 7 has a considerable influence on the final results.

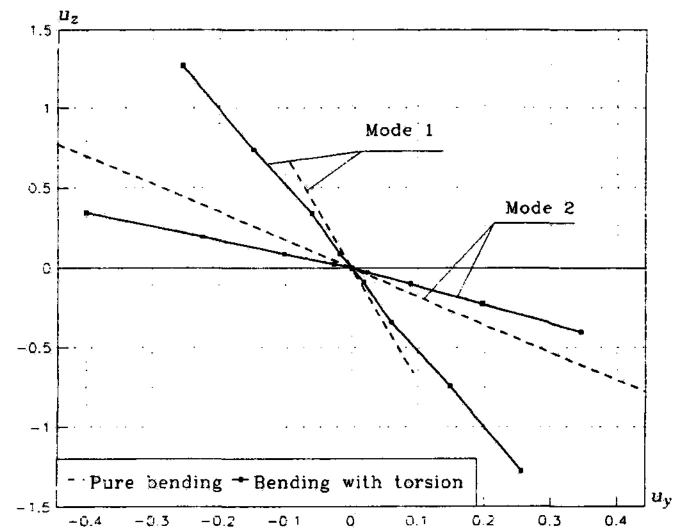

Fig. 7. Trajectories of motion of free-end point 1

6. Concluding Remarks

On the basis of semi-analytical theory evolved, a novel type of semi-analytically based finite elements is derived. The SABFE method proposed possesses some significant advantages in comparison with the classical beam theories:

a) Nodal displacements are compatible with the displacements of a three-dimensional body;

b) The method offers extended possibilities for introducing more complex cases loading and supports.

The SABFE elements are tested through a series of numerical examples and through a comparison with other solutions so that the essential properties of SABFE method could be confirmed. It is clear, however, that a comprehensive analysis and extensive future research are required in order to develop and extend the SABFE method.

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PUSIAUANALIZINĖS TEORIJOS BAZĖJE SUFORMULUOTI BAIGTINIAI ELEMENTAI IR JŲ TAIKYMAS PLONASIENĖMS SIJOMS

R. Kačianauskas

S a n t r a u k a

Pateikiami naujo tipo pusiauanalizinės teorijos pagalba suformuluoti sijų baigtiniai elementai. Šie elementai išvedami iš pusiauanalizinės aukštesnės eilės sijų teorijos, nagrinėjančios sijos skerspjūvį kaip pusiauanalizinių elementų ansamblį. Elemento modelis yra sudaromas prisilaikant hierarchinės dviejų pakopų schemos, kur išskiriami pirminiai ir antriniai kintamieji. Aptartos abiejų pakopų funkcinės (pusiaudiskretinės) priklausomybės. Jos diskretizuojamos mišriais baigtiniais elementais, panaudojant Lagranžo daugiklių metodą. Bendrosios baigtinio elemento priklausomybės panaudotos plonasienių sijų elementams sudaryti. Gauti elementai išbandyti skaitiškai sprendžiant statikos ir dinamikos uždavinius bei lyginant su kitais sprendiniais.

Pasiūlyti elementai turi laisvės laipsnius, suderinamus su kontinualiais kūnais, bei leidžia spręsti uždavinius, kur apkrovų bei atramų išsidėstymas išeina iš klasikinių sijų teorijų rėmų.